USING LOCAL SURROGATE INFORMATION IN LAGRANGEAN RELAXATION: AN APPLICATION TO SYMMETRIC TRAVELING SALESMAN PROBLEMS

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The *Traveling Salesman Problem (TSP)* is one of the most studied problems in Combinatorial Optimization literature. Several articles have been published on the subject and it remains today as an interesting and challenging problem. The most common interpretation of the problem seeks the shortest tour for a salesman on a number of cities or clients. Clients must be visited exactly one time and the salesman must return to their home city. For a comprehensive survey of solution methods, applications and related problems see the book of Lawler et al. [27]. Laporte [25] gives another review, including applications examples on computer wiring, wallpaper cutting, hole punching, job sequencing, dartboard design and crystallography. The problem is well known to be NP-hard [25] justifying the use of heuristics, mainly for large scale problems. Johnson and McGeoch [20] give a recent survey on the use of local search based heuristics.

Lagrangean relaxation is a well known relaxation technique frequently used to give bound information to combinatorial optimization problems [see for example the survey papers [9, 10] and the books [32, 36]). Held and Karp [17, 18] applied Lagrangean relaxation to *TSP* in beginning of seventies. The relaxation limit approximates what is known today as *HK* (Held and Karp) bound, a very good bound (less than 1% from optimal) for a large class of symmetric instances [21]. Johnson et al. [21] report that exact *HK* bounds have been computed by a special purposed linear programming code, for instances as large as 33,810 cities. For even large instances, is applied the subgradient method proposed on the original Held and Karp papers and speeded up by a number of algorithmic tricks [2,16,34,37,38]. Since for large instances we do not know the optimal solution, the comparison of the heuristic and *HK* bounds is common practice.

Although of simple convergence conditions [8, 33], the convergence of subgradient methods can consume a long time for some instances. The subgradient optimization is very sensitive to the initial values for the multipliers and the rules applied for step size controlling. Efforts were made to have theoretical foundations for these choices [3, 13], but until today the most popular

approaches are based on previous empirical experience [19]. Other subgradient methods appeared in literature [4,5,6,23,24,26]. More elaborated, they increase the local computational times computing descent directions [6], or combining subgradients of previous iterations [4,5], or realizing projections onto general convex sets [23,24,26]. Experimental results with some of these methods show an improvement in performance compared to the subgradient method [23,26]. The subgradient method remains the widely used approach in the Lagrangean relaxation context.

Reducing the initial erratic behavior of the subgradient method can result in fast convergence. For large scale problems that can be interesting, even with the use of fast computers. The Lagrangean relaxation can be adapted to use local information (optimization) provided by the surrogate constraints to accelerate the subgradient method, conserving the same *HK* bounds. The idea is to introduce a local optimization step at the initial iterations of a subgradient method. The first relaxation is a surrogate relaxation of the assignment constraints at the *TSP* formulation, followed by a Lagrangean relaxation of the surrogate constraint. A local Lagrangean dual optimization is approximately solved. The process is repeated for a pre-defined number of iterations of the subgradient method.

The surrogate duality theory is an old matter, that was not so intensively explored as the Lagrangean counterpart (see the papers [7, 11, 12, 14, 22] and the book [32] for a formal view of the subject). We explore here the simple relationship between the two relaxations, recalling that Lagrangean multipliers can also be considered surrogate multipliers, and making profit of the local optimization proportioned by the new local Lagrangean relaxation. The combination of the two relaxations is known as a *Lagrangean/surrogate* relaxation, and is best described on the Narciso and Lorena work [31], for an application to Generalized Assignment problems. It appears to be indicated to improve the use of subgradient schemes on Lagragean relaxation context, at least the less elaborated ones as the traditional subgradient method.

Consider a *TSP* defined on a graph $G = (V,E)$, $V=\{1,...,n\}$, and let the binary variable x_{ij} be equal to 1 if the edge (i,j) $\hat{I}$ E is used in the optimal tour. $C = [c_{ij}]$, where $c_{ij} = c_{ji}$ for all i,j $\hat{I}$ V, is a distance (or cost) matrix associated with E. The formulation is

$$\text{(P)} \qquad \begin{aligned} & Min \sum_{i<j} c_{ij} x_{ij} \\ \text{subject to } & \sum_{i<k} x_{ik} + \sum_{j>k} x_{kj} = 2, \quad k = 1,\dots,n, && (1) \\ & \sum_{i,\, j \in S} x_{ij} \le |S| - 1, \quad S \subset V, \quad 3 \le |S| \le n-3, && (2) \\ & x_{ij} \in \{0,1\}, \; i, j = 1,\dots,n, \quad i < j. && (3) \end{aligned}$$

Constraints (1) specify that every vertex has degree 2, constraints (2) are subtour elimination constraints, and (3) the binary conditions.

The traditional multipliers l_k, $k \in V$, can be seen as surrogate multipliers, and $\sum_{k \in V} l_k \left(\sum_{i<k} x_{ik} + \sum_{j>k} x_{kj} - 2 \right) = 0$, as a surrogate constraint. Using a one-dimensional multiplier $t \hat{I} R$, and relaxing this surrogate constraint in the Lagrangean way gives the surrogate version of the Lagrangean function (named *Lagrangean/surrogate* in [31])

$$L_t(l) = Min_x \left\{ \sum_{i<j} c_{ij} x_{ij} + \sum_{k \in V} t.\, l_k \left(\sum_{i<k} x_{ik} + \sum_{j>k} x_{kj} - 2 \right) \right\},$$

where x is a feasible solution to the length of the 1-spanning tree, obtained by the shortest tree having vertex set V\{1} and two minimal distinct edges at vertex 1.

In our notation $L(l) = L_1(l)$ is the usual Lagrangean relaxation. The Lagrangean bound is improved searching the solution of the dual $D(l) = Max_l \{L_t(l)\}$ (the dual value is the same for both relaxations). For a given l, a local dual can be identified here as $D_t(l) = Max_t \{L_t(l)\}$. It is interesting to note that for $t = 1$ the local optimization induced by the surrogate constraints is not considered. The same for each fixed value of t. Also is immediate that for the same l, $v[D_t(l)] \geq v[L(l)]$, i.e., the local dual gives an improved bound to the usual Lagrangean relaxation (v[(.)] is an optimal value for problem (.)). The local dual is approximately solved by a naive one dimensional search for the best parameter t.

The subgradient method is employed to solve problem $D(l)$, giving an approximated *HK* bound for problem *(P)*. We propose here to use the traditional subgradient method, with the step size corrections provided by Held and Karp [18], without any modification or improvement. That decision will respond the question if the original HK step was a good one. Observing the literature for other suggestions on step size corrections and/or new step sizes, it become evident the necessity of such modifications [3, 5, 16, 21, 34, 37, 38].

The multiplier updates observe the following formula

$$l^{i+1} = l^i + b\,[v_f - v(L_t(l^i))] g_t^{l^i} / \left\| g_t^{l^i} \right\|^2, \quad 0 \leq b \leq 2 \qquad (4)$$

(where v_f is the value of a feasible solution to (P)). It is easy to see different relaxation bounds sequences observing that the subgradients are distinct, $g_t^l \neq g_1^l$ (in general). The control parameter b is the Held and Karp [19] control that makes $0 \leq b \leq 2$, beginning with $b = 2$. If after 20 iterations $v[L_t(l)]$ not increases, b is updated to $b = b/2$.

A sample of symmetric instances was selected from the TSPLIB (http://www.iwr.uni-heidelberg.de/iwr/comopt/soft/TSPLIB95/tsp) to conduct a computational comparison between the application of the usual Lagrangean relaxation (multiplier t is fixed to 1 at each iteration of the subgradient method) and the Lagrangean/surrogate (that explore the one dimensional search for t on a number of initial iterations of the subgradient method).

The instances are: *uly**16**m; uly**22**m; att**48**; berlin**52**; kroA**100**; tsp**225**; pcb**442**; pr**1002**; d**1291**, rl**1304**; nrw**1379**; d**1655**; vm**1748**; rl**1889** and u**2152*** (refereed in the following simply as ***16**, **22**, **48**, **52**, **100**, **225**, **442**, **1002**, **1291**, **1304**, **1379**, **1655**, **1748**, **1889 and 2152**).*

Let *gap = (optimal solution – relaxation)/optimal solution. Table 1* shows for each problem *the Lagrangean best gap (%) and the elapsed time (time1) to reach this gap, the Lagrangean/surrogate best gap (%) and the elapsed time (time2 to reach the Lagrangean best gap, and finally the time ratios time2/time1 (%).* The Lagrangean/surrogate was able to reach 6 tighter bounds, all after the size of *1002*. Observing the last column we can see, for example, that it reached the better Lagrangean bound using only 2.6 % of time on problem *1889* , 2.8% on problem *1002*, 3.7% on problem *1748*, 5.3% on problem *1304*. The economy of time was not representative for two small instances (*52* and *225*).

We investigated in this paper the effects of local search on Lagrangean relaxation applied to symmetric TSP. The local search was simply justified considering the Lagrangean multipliers as surrogate multipliers, affected by a local one-dimensional Lagrangean dual. The local search can be a straight one, giving in few iterations a better one-dimensional multiplier than the usual Lagrangean multiplier (fixed in one). We hope that the Lagrangean/surrogate approach can be useful for even large scale TSP instances, considering the importance of *HK* bounds for heuristic performance comparison [20, 21]. It is also important to note that the refereed approach is independent of the step size and subgradient direction used (if the convergence conditions were observed).

Problem	*Lagrangean best gap %*	*Time1 (sec) Lagrangean best gap*	*Lagrangean/ surrogate best gap %*	*Time2 (sec) Lagrangean/surrogate reaches Lagrangean best gap*	$\frac{Time2}{Time1}\%$
16	0.1	2.	0.1	1.03	51
22	0.1	9.1	0.1	4.6	51
48	0.3	19.	0.3	8.	42
52	0.3	5.	0.3	5.	100
100	2.	27.	2.	14.	51
225	4.	495.	4.	392.	92
442	1.	4054.	1.	997.	24
1002	4.	36714.	2.	1054.8	2.8
1291	3.	13431.	3.	3230.	24
1304	5.	28094.3	2.	1511.	5.3
1379	2.	9465.7	2.	3147.	33
1655	3.	29368.	2.	3029.	10
1748	5.	48413.	2.	1802.	3.7
1889	5.	87568.	2.	2275.4	2.6
2152	2.	31334.	1.	3648.	11.6

Table 1: Comparison: Lagrangean versus Lagrangean/surrogate

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References

[1] Almiñana, Marcos e Pastor, T. J. "An adaptation of SH heuristic to the location set covering problem". European Journal of Operational Research, 100, 586-593, 1997.
[2] Allen, E., Helgason, R., Kennington, et al. "A generalization of Poliak's convergence results for subgradient optimization". Mathematical Programming, 37, 309-317, 1987.
[3] Bazaraa, M. S., Sherali, H. D. "On the choice of step size in subgradient optimization". European Journal of Operational Research, 7, 380-388, 1981.
[4] Brännlund, U., "A Generalized subgradient method with relaxation step", *Mathematical Programming* 71 (1995) 207-219.
[5] Camerini, P. ; Fratta, L. and Maffioli F., "On improving relaxation methods by modified gradient techniques" *Mathematical Programming Study*, 3, (1975) 26-34.
[6] Correa, R. and Lemaréchal, C, "Convergence of some algorithms for convex minimization", *Mathematical Programming* 62 (1993) 261-275.
[7] Dyer, M. E. "Calculating surrogate constraints". Mathematical Programming, 19, 255-278, 1980.
[8] Ermol'ev, Y. M. "Methods for solving nonlinear extremal problems". Cybernetics, 16/1, 1-14, 1966.
[9] Fisher, M. L. "The lagrangian relaxation method of solving integer programming problems". Management Science, 27, 1-18, 1981.
[10] Geoffrion, A. " Lagrangean relaxation and its uses in integer programming". Mathematical Programming Study, 2, 82-114, 1974.
[11] Glover, F. "Surrogate constraints". Operations Research, 16(4):741-749, 1968.
[12] Glover, F. "Surrogate Constraints Duality in Mathematical Programming". Operations Research, 23, 434-451, 1975.
[13] Goffin, J. L. "On convergence rates of subgradient optimization methods", Mathematical Programming, 13, 329-347, 1977.
[14] Greenberg, H. J., Pierskalla, W. P. "Surrogate Mathematical Programming". Operations Research, 18, 924-939, 1970.
[15] Handler, G., Zang, I. "A dual algorithm for the constrained shortest path problem". Networks, 10, 193-310, 1980.
[16] Helbig-Hansen, K. H., Krarup, J. "Improvements of the Held-Karp algorithm for the symmetric traveling salesman problem". Mathematical Programming, 7, 87-96, 1974.
[17] Held, M., Karp, R. M. " The Traveling salesman problem and minimum spanning trees". Operations Research, 18,1138-1162, (1970).
[18] Held, M., Karp, R. M. " The Traveling salesman problem and minimum spanning trees: Part II". Mathematical Programming 1, 6-25, (1971).
[19] Held. M., Wolfe, P., Crowder, H. P. "Validation of subgradient optimization", Mathematical Programming, 6, 62-88, 1974.
[20] Jonhson, D.S., McGeoch, L.A. "The traveling salesman problem: a case study in local optimization". In *Local search in Combinatorial optimization*, E. H. L. Aarts and J. K. Lenstra (eds.), John Wiley & Sons, New York, 1997.
[21] Jonhson, D.S., McGeoch, L.A., Rothberg, E. E. "Asymptotic Experimental Analysis for the Held-Karp Traveling Salesman Bound. Proceedings of the7th Annual ACM-SIAM Symposium on Discrete Algorithms, 341-350, 1996.
[22] Karwan, M. H., Rardin, R. L. "Some relationships between lagrangian and surrogate duality in integer programming". Mathematical Programming, 17, 320-334, 1979.
[23] Kim, S. and Um, B. S., "Polyak's subgradient method with simplified projection for nondifferentiable optimization with linear constraints", *Optimization* 20 (1989) 451-456.
[24] Kim, S. and Um, B. S., "An improved subgradient method for constrained nondifferentiable optimization", *Operations Research Letters* 14 (1993) 61-64.
[25] Laporte, G. "The traveling salesman problem: an overview of exact and approximate algorithms". European Journal of Operational Research, 59, 231-247, 1992.
[26] Larsson, T. ; Patriksson, M. and Strömberg, A-B., "Conditional subgradient optimization - theory and applications", *European Journal of Operational Research* 88, 382-403, 1996.
[27] Lawler, E. L. ; Lenstra, J. K. ; Rinnooy Kan, A. H. G. and Shmoys, D. B., "The traveling salesman problem, John Wiley and Sons, Chichester, 1985.
[28] Lorena, L. A. N., Lopes, F. B. " A surrogate heuristic for set covering problems". European Journal of Operational Research. 79(1), 138-150, 1994.
[29] Lorena, L. A. N., Narciso, M. G. "Relaxation heuristics for a generalized assignment problem". European Journal of Operational Research, 91(1), 600-610, 1996.
[30] Minoux, M.. "Plus courts chemins avec constraints: Algorithmes et applications", Annals of Telecommunications, 30, 383-394, 1975.
[31] Narciso, M. G., Lorena, L. A.N. "Lagrangean/surrogate relaxation for generalized assignment problems". European Journal of Operational Research,114(1), 165-177, 1999.
[32] Parker, R. G., Rardin, L. R. "Discrete Optimization". Academic Press, INC, London, 1988.
[33] Poljak, B. T. "Minimization of unsmooth functionals". USSR Computational Mathematics and Mathematical Physics 9, 14-29, 1969.
[34] Reinelt. G. "The traveling salesman problem: computational solutions for TSP applications". Lecture Notes in Computer Science 840, Springer Verlag, Berlin, 1994.
[35] Senne, E.L.F. and Lorena, L.A.N "A lagrangean/surrogate approach to facility location problems", EURO-TIMS Congress – Barcelona, 1997.
[36] Shapiro, J. F. "Generalized lagrange multipliers in integer programming". Operations Research, 19, 68-76, 1971.
[37] Volgenant, T., Jonker, R. "A branch and bound algorithm for the symmetric traveling salesman problem based on the 1-tree relaxation". European Journal of Operational Research, 9, 83-89, 1982.
[38] Valenzuela, C. L., Jones, A J. "Estimating Held-Karp lower bond for the geometric TSP", 1995.