Controlling Complexity

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How do we know that a system dynamics is complex ?

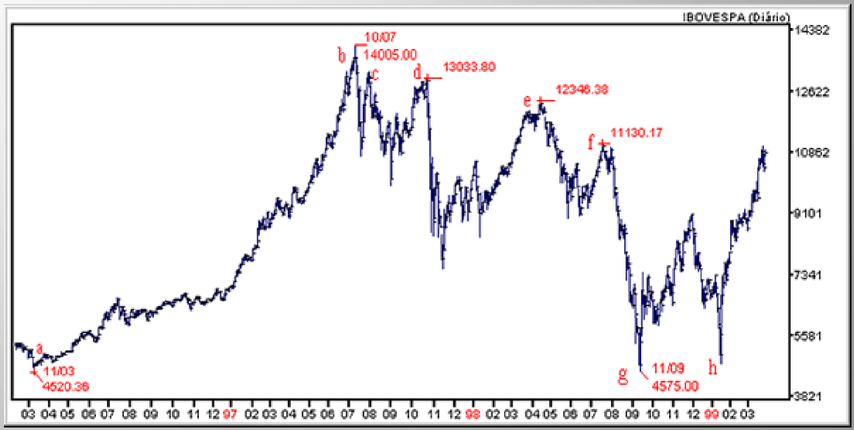
- It is a fundamental question !

- It may present a “complex behavior”...

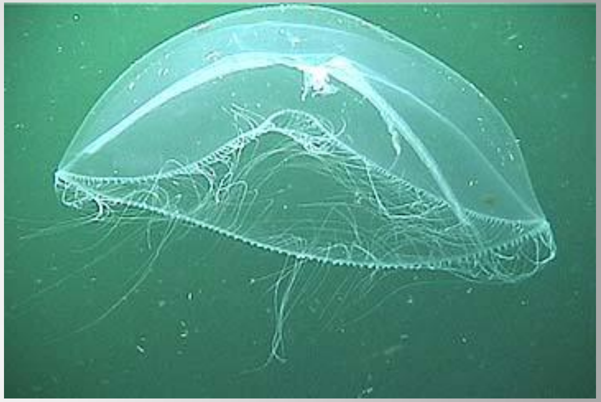
- We “see” it and we have the feeling that it is “complex”…

System with a *“Complex”* behavior I

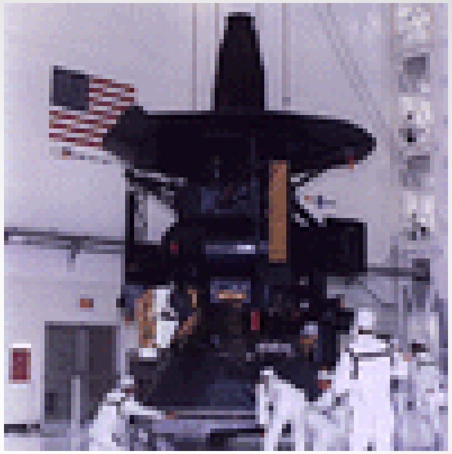
- Many systems that surround us are “*Complex”:*



System with a *"Complex"* behavior II



Man made Systems with a *“Complex”* behavior



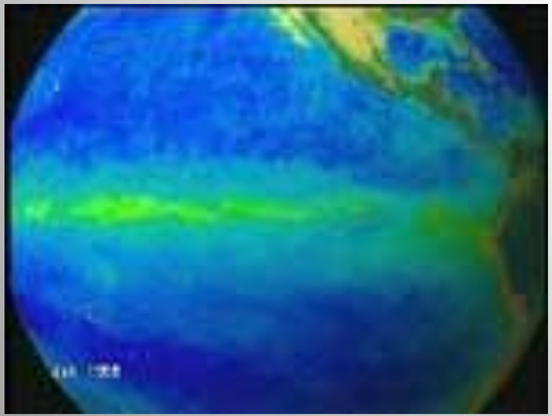
Characteristics of a Complex System

- Regarding the system's behavior, if it is a *Complex System*, we might expect to find the following:

1) A behavior that is neither completely ordered and predictable nor completely random and unpredictable;
2) Its evolution reveals patterns in which *coherent structures* develop at various scales, but do not exhibit elementary interconnections;
3) The structures can show a *hierarchical relationship*, i.e., nontrivial structures over a wide range of scales can appear.

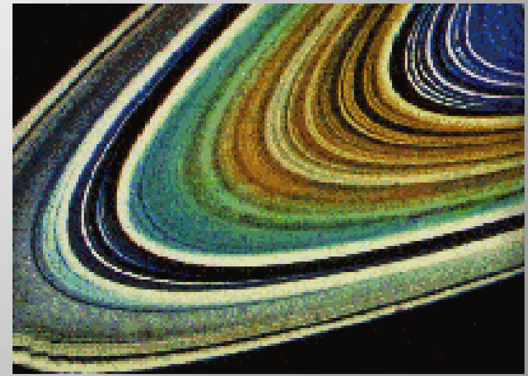
Complex Systems and interdependent parts

- System that has a *global emergent property* can be identify as being formed out of *interdependent parts*.
- *Interdependent*: the influence one part has on another.
- *Interdependent* is distinct from "*interacting*", because even strong interactions do not necessarily imply interdependence of behavior (ex: macroscopic properties of solids).
- *Collective behavior* result from the *interdependency* of parts.



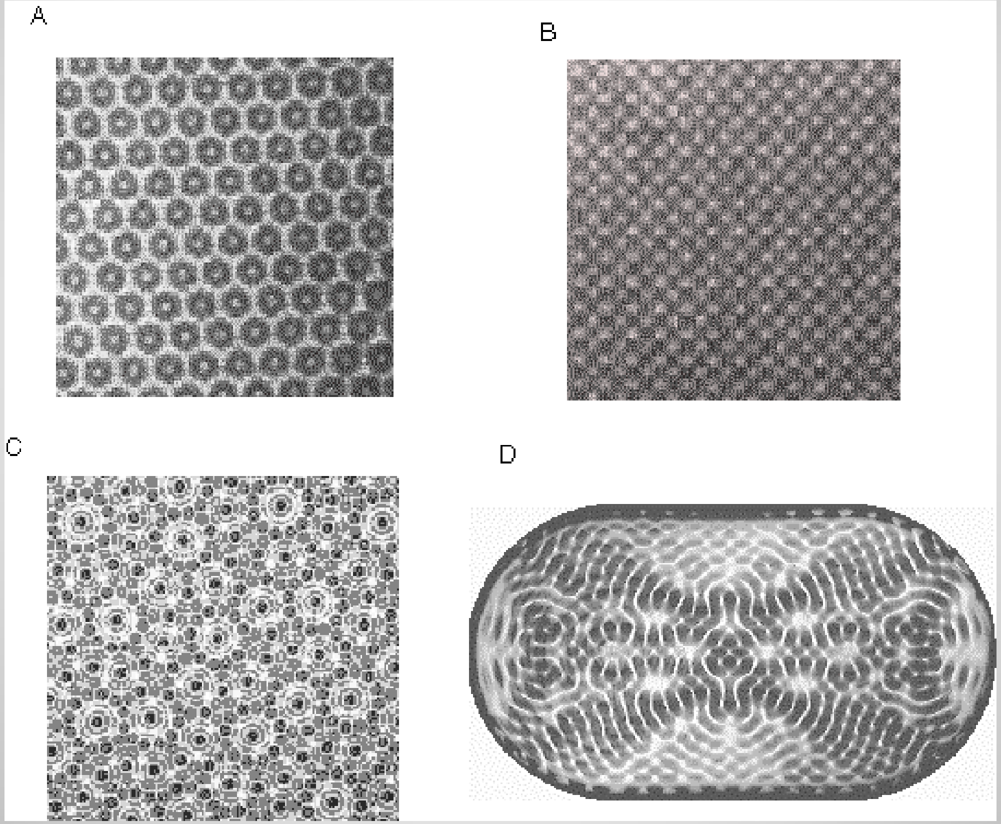
El nino

Galileo



Saturn rings

Examples of Complex Systems 1

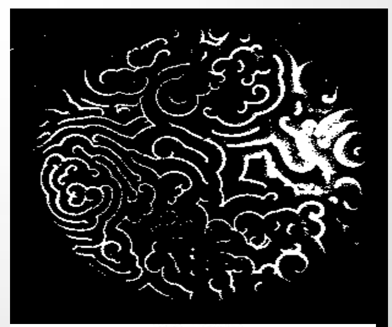


Patterns of standing waves on fluid surfaces generated by vibrating the containing vessel with various driving frequencies

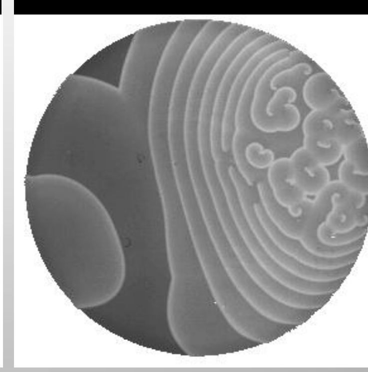
Examples of Complex Systems

- Many chemical reactions exhibit oscillations. An oscillation is a repetitive wave that passes through zero - delineating a transition through two distinct states (+ve and -ve).
- The Belousov-Zhabotinsky reaction is a visual reaction between waves of oxidation and reduction that show color changes to represent phase changes.



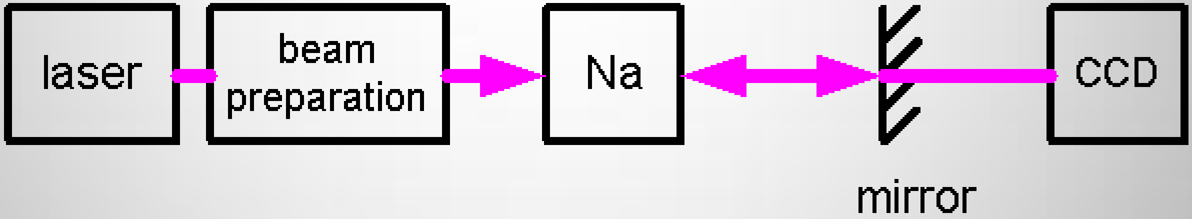


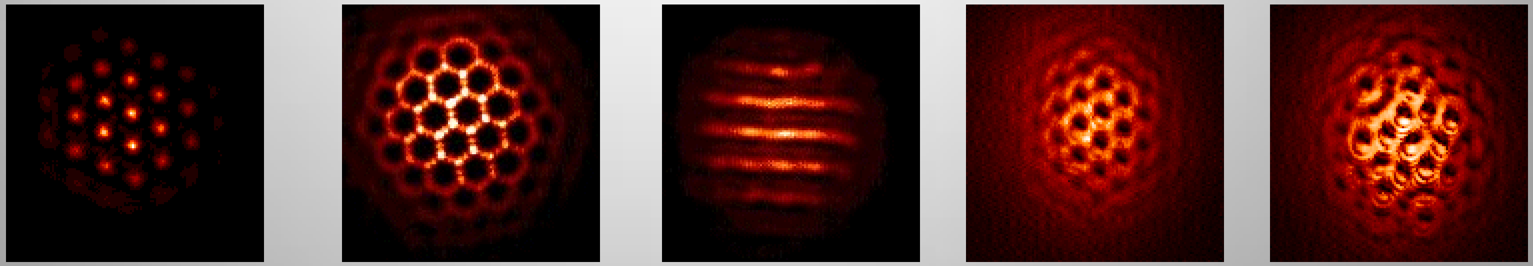





Examples of Complex Systems

- A wide range of complex phenomena can be observed in nonlinear optics: temporal instabilities, disordered patterns, spontaneous formation of structures and vortices;
- The basic mechanism: nonlinear interaction between electromagnetic waves and atomic medium $\Rightarrow$ excitation of many modes;





Complex system characteristics typically appear in...

- *Systems with many degrees of freedom;*
- For these systems we have a situation where a large number of booth attracting and unstable chaotic sets coexist.
- As a result, we can have a rich and varied dynamical behavior, where many competing behaviors can exist.
- System evolving in the neighborhood of an attracting periodic set $\Rightarrow$ "*ordered*" behavior;
- System evolving about the unstable sets $\Rightarrow$ "*non-ordered*" behavior;
- The behavior keep changing from one behavior to another, as the system evolves.

Complexity in Low Dimensional Systems

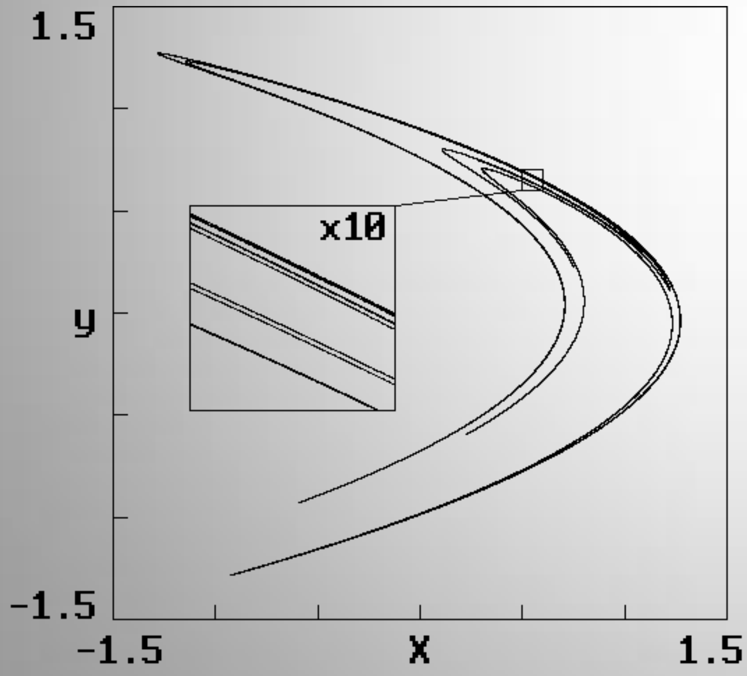
- A complex behavior can also appear in *low dimensional systems*!
- Low dimensional systems with large number of coexisting periodic attractors and a complicated fractal basin structures can present a complex behavior :
  - Double rotor with noise [PRL **75**/4023]
  - Single rotor with noise [Chaos **7**/597].
- *Multistability:* the key to understand how the complexity thrives in low dimensional systems.

Multistability:

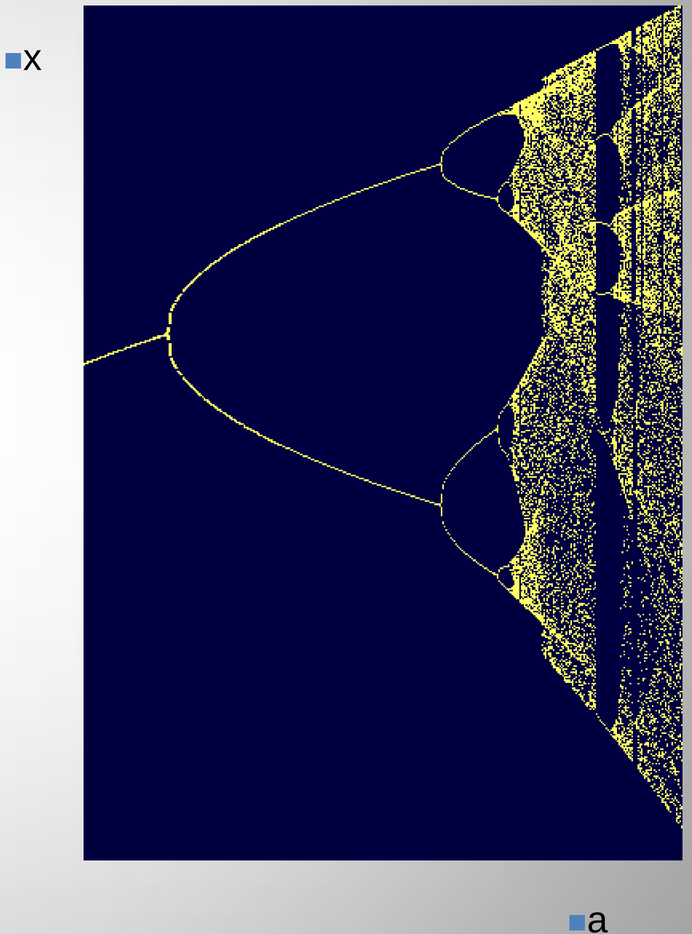
- Multistability means the coexistence of several final states (attractors) for a given set of parameters.
- The long-term behavior of such systems becomes more involved, because there exists a *nontrivial relationship* between these coexisting asymptotic states and their basins of attraction.
- Mtulitstable behavior is found in
  - semiconductor physics;
  - chemistry
  - neuroscience
  - laser physics
  - ...

Hénon attractor

- 2-D map given by the equations:
  - $x_{n+1} = y_n + a - bx_n^2$
  - $y_{n+1} = x_n$



a = 1.4 b=0.3

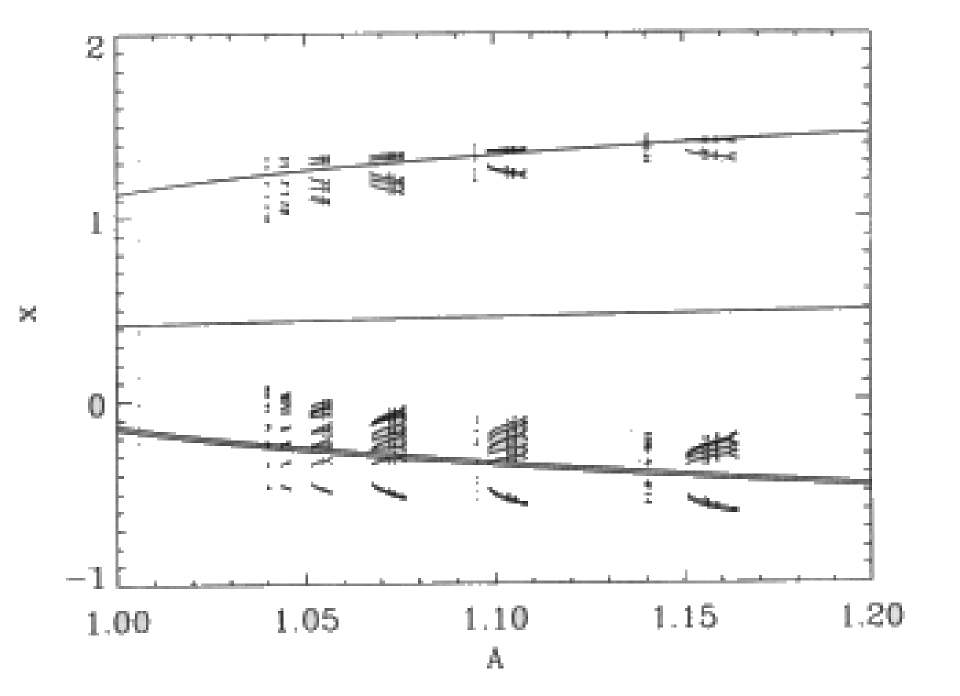


Hénon map with “small” amount of damping

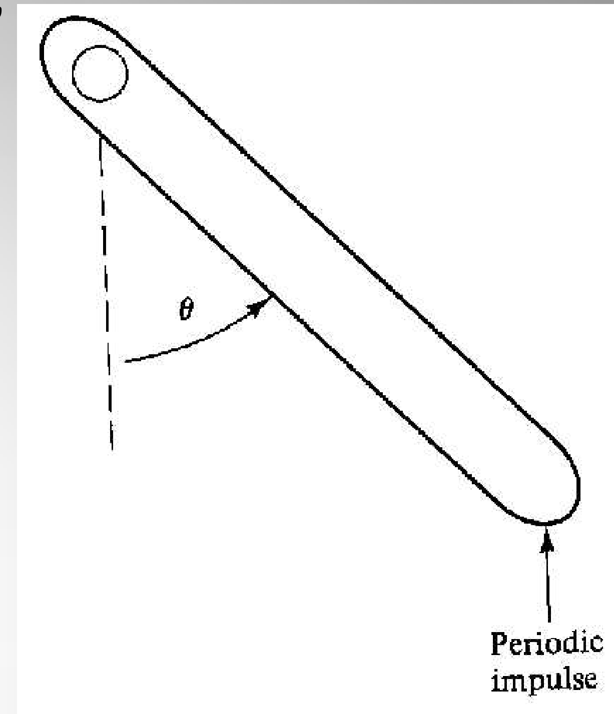
- A = bifurcation parameter
- $v \in [0,1] \leftarrow$ “dissipation”
- v =0 $\Rightarrow$ Jacobian matrix =1 $\Rightarrow$ map is conservative.
- v = 1 $\Rightarrow$ equations are decoupled $\Rightarrow$ quadratic map.
- v “small” (~0) $\Rightarrow$ there are several coexisting attractors!

$$x_{n+1} = A - x_n^2 - (1-\nu)\, y_n$$

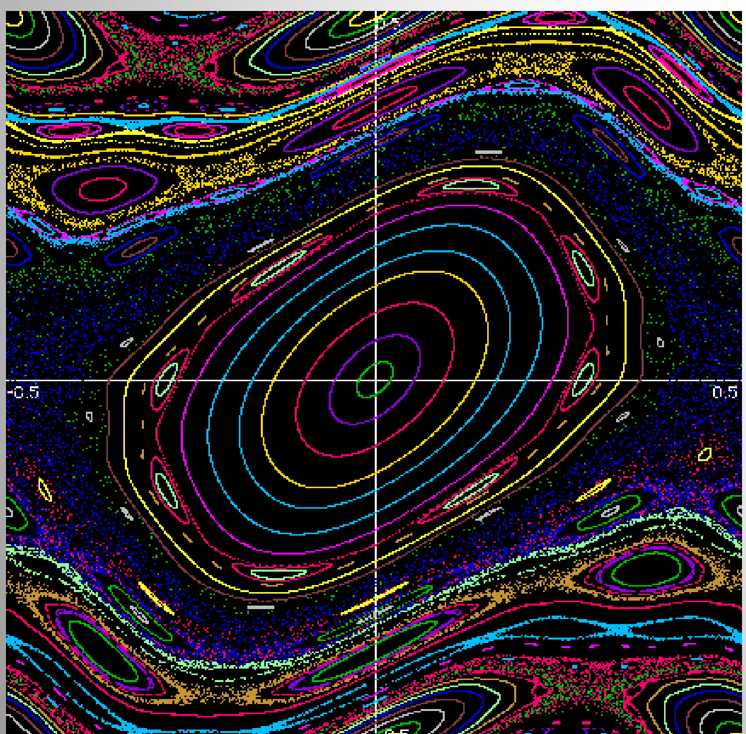
$$y_{n+1} = x_n\,.$$



“kicked single rotor”



- No damping case (v=0): area-preserving *standard map*;
- It has stable and unstable periodic orbits, KAM surfaces and chaotic regions.



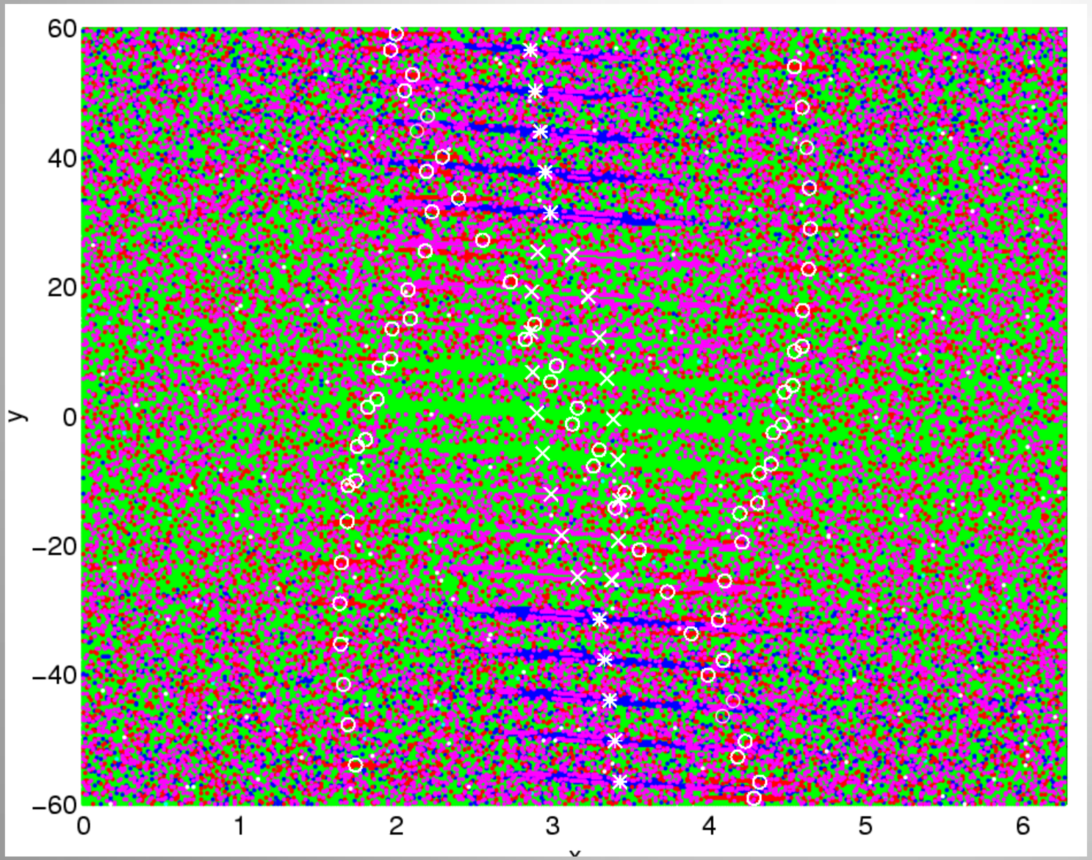
$$x_{k+1} = (x_k + y_k) \bmod 2\pi$$

$$y_{k+1} = (1-v)y_k + f_0 sin(x_k + y_k)$$

- f_0: force parameter;
- v: damping parameter;
- Dynamics lies on the circle $[0,2\pi)$

Single Rotor with "Small" Dissipation

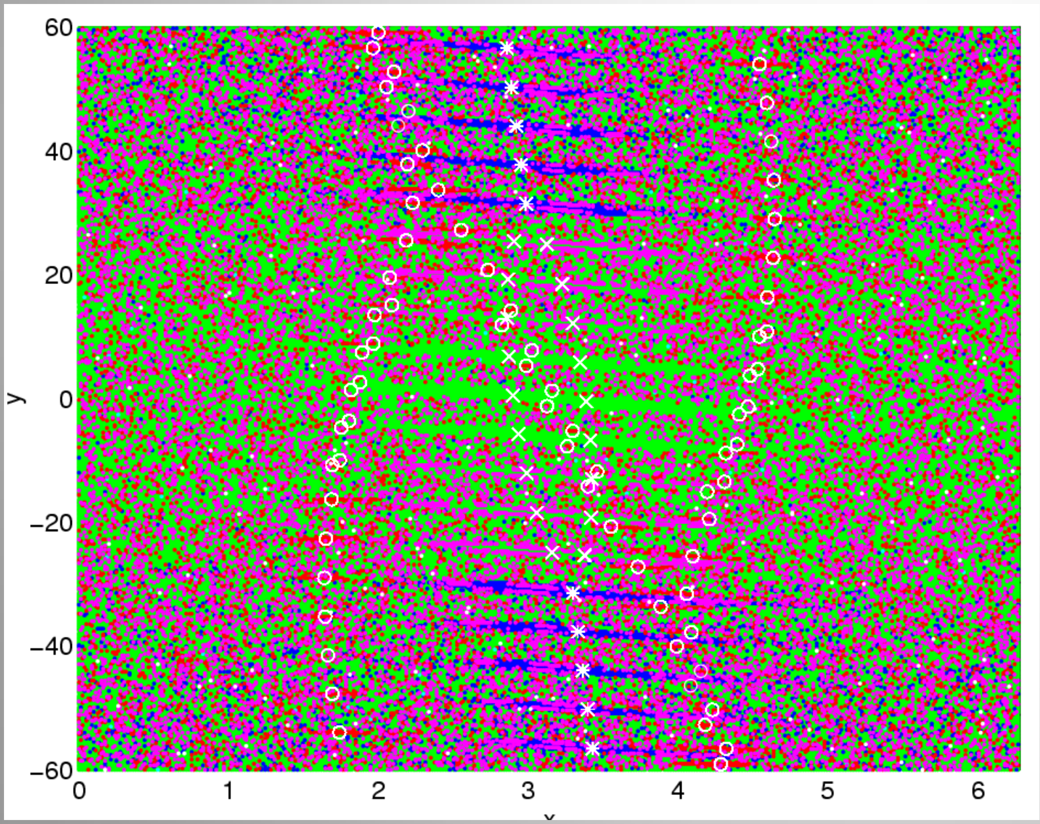
$$x_{k+1} = (x_k + y_k)\mathrm{mod}\ 2\pi$$
$$y_{k+1} = (1-\nu)y_k + f_0 sin(x_k + y_k)$$



- For $\nu \approx 0$ (**very small amount of dissipation**):
  - The symmetry in *y* is broken;
  - The motion takes place on the cylinder $[0,2\pi)\times\Re$;
  - Periodic orbits become sinks;
  - The dissipation leads to a separation of the overlapping periodic orbits, which belongs to a given family, with increasing module of the velocities on the cylinder.

Single Rotor with "Small" Dissipation

$$x_{k+1} = (x_k + y_k) \bmod 2\pi$$
$$y_{k+1} = (1 - v) y_k + f_0 sin(x_k + y_k)$$



Basin of attraction

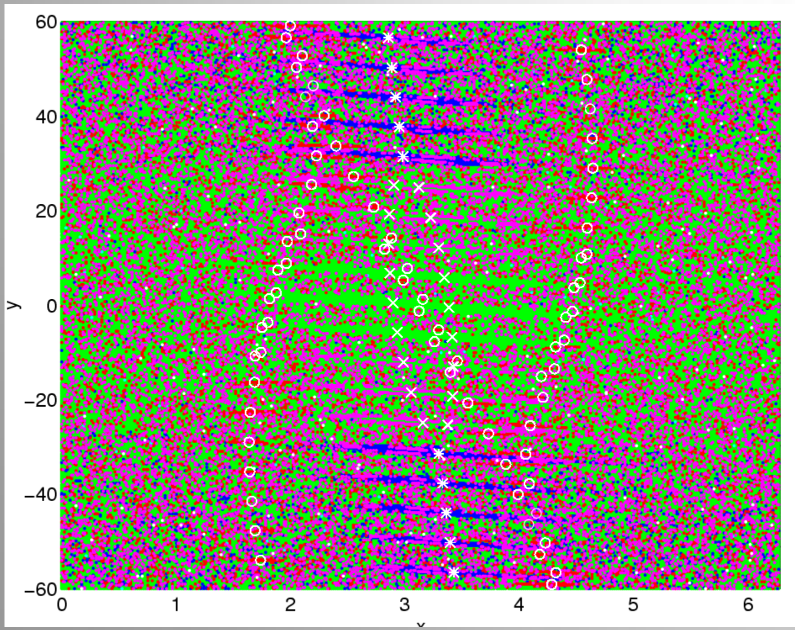
- For $v \approx 0$ (**very small amount of dissipation**):
  - Great number of coexisting attracting periodic orbits of increasing period;
  - There is a bounded cylinder $[0, 2\pi) \times [-y_{max}, y_{max}]$, where $y_{max} = f_0/\nu$ which contains all of the attractor;
  - All trajectories are eventually trapped inside.

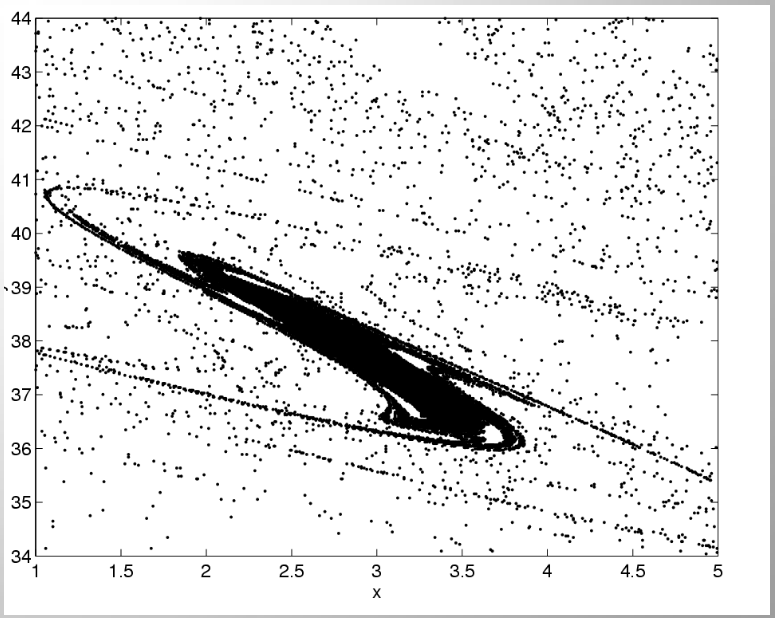
Single Rotor with "Small" Dissipation

$$x_{k+1} = (x_k + y_k) \bmod 2\pi$$

$$y_{k+1} = (1-\nu)y_k + f_0 sin(x_k + y_k)$$

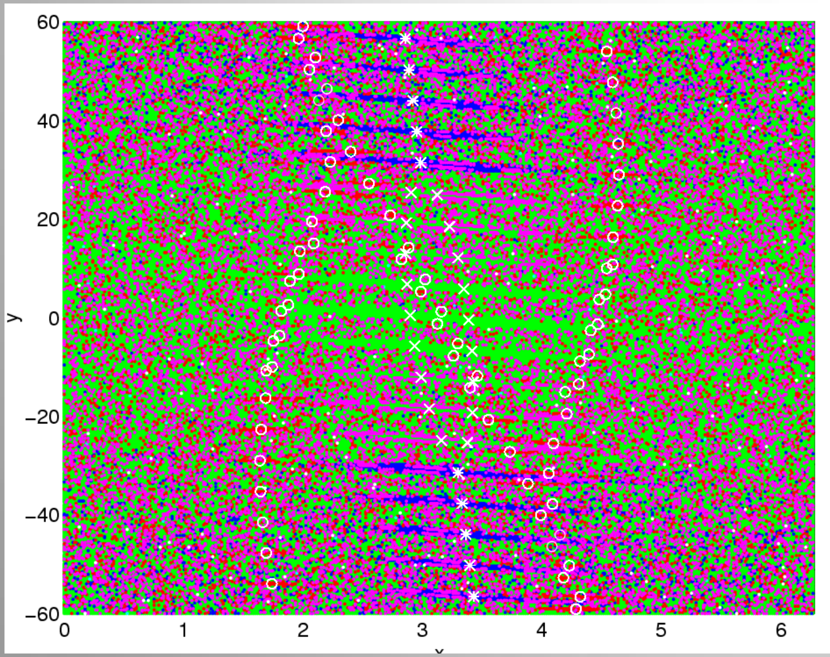
- For $\nu \approx 0$ (**very small amount of dissipation**):
  - The basin of attraction for the periodic orbits have fractal basin boundaries;
  - Basin boundaries are organized in a complex interwoven structure that permeate most of the state space, with chaotic saddles embedded in them.
  - Box counting dimension d = 1.994.

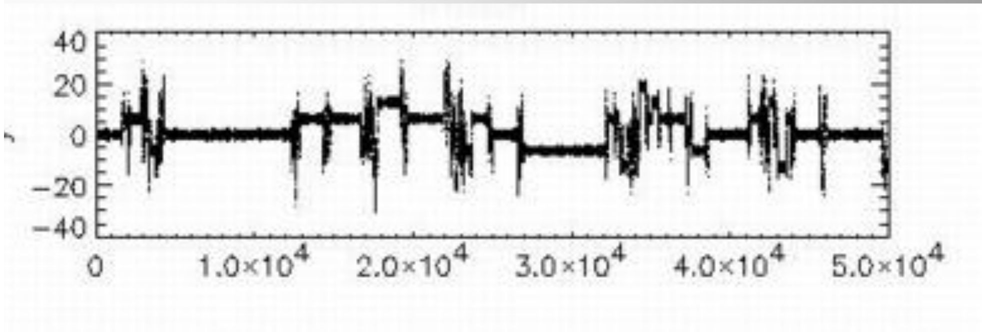




Single Rotor with "Small" Dissipation

- For $\nu \approx 0$ (**very small amount of dissipation**):
  - Basin boundaries are organized in a complex interwoven structure that permeate most of the state space, with chaotic saddles embedded in them.
  - Chaotic sets become unstable chaotic sets embedded in the basin boundaries separating the various sinks;
  - Chaotic motion is replaced by long *chaotic transients* that occur before the trajectory is eventually asymptotic to one of the sinks.
  - High sensitive to the final state $\Rightarrow$ *multistability* !

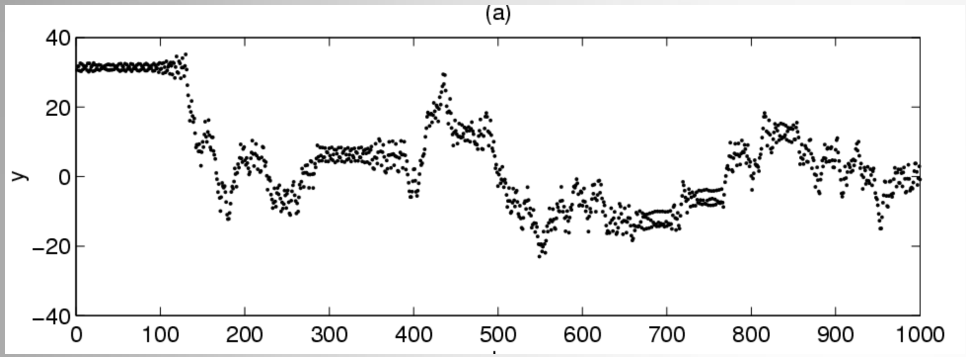


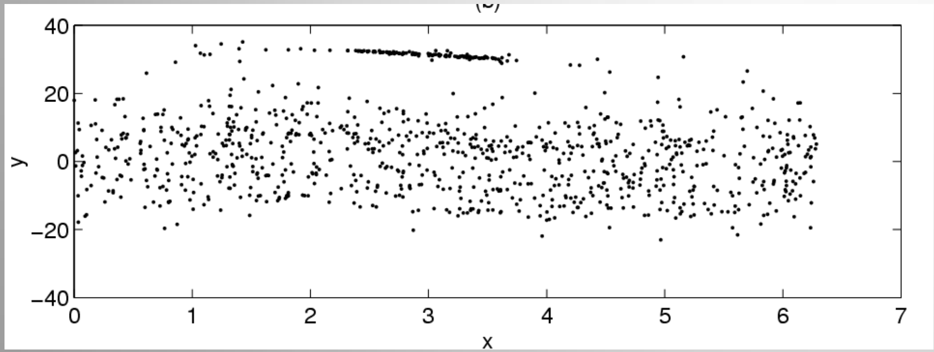


Hopping dynamics for a periodic kick rotor with small dissipation

From Multistability to Complexity

$$x_{k+1} = (x_k + y_k) \mathrm{mod}\ 2\pi + \delta$$
$$y_{k+1} = (1 - v) y_k + f_0 sin(x_k + y_k) + \delta$$
$$\delta = small\ amplitude\ noise$$

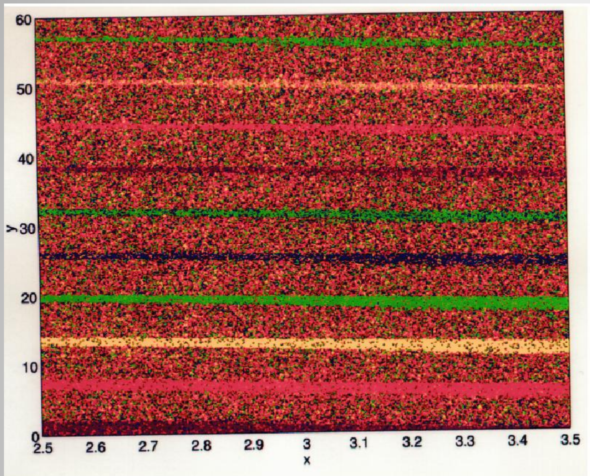




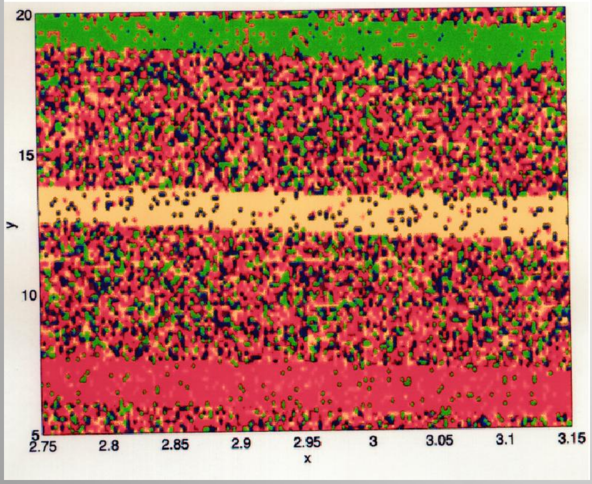
- The noise may prevent the trajectories from settling into stable periodic behavior;
- Trajectories with long chaotic transients $\Rightarrow$ “random” like behavior;
- Trajectory may come close to one of the periodic attractors and remain in its neighborhood $\Rightarrow$ “ordered” behavior;
- Noise will eventually move the trajectory out of the “ordered” behavior into the fractal boundary region $\Rightarrow$ chaotic transient.

Coherent structures

$$x_{k+1} = (x_k + y_k) \bmod 2\pi + \delta$$
$$y_{k+1} = (1-v)y_k + f_0 sin(x_k + y_k) + \delta$$
$$\delta = small\ amplitude\ noise$$



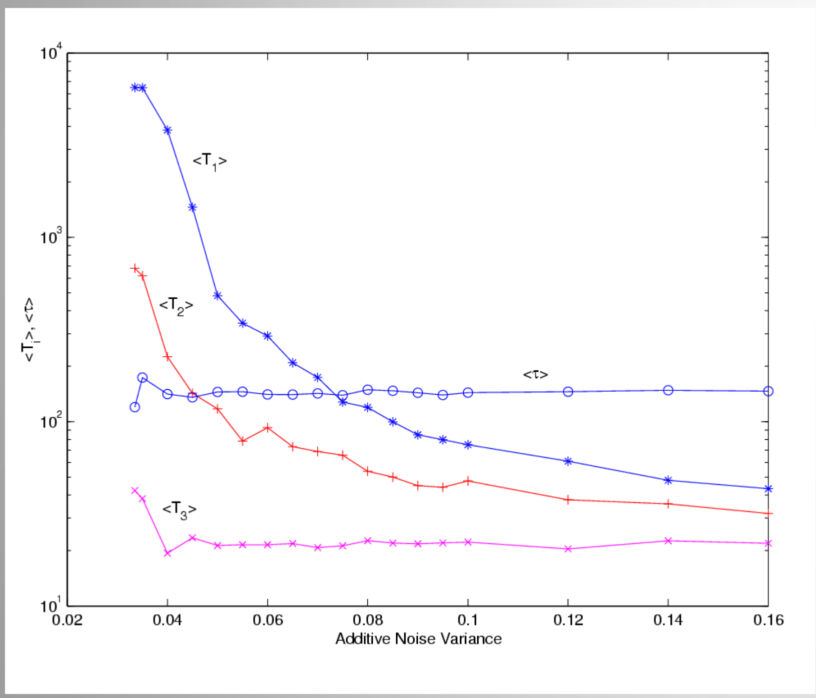
(A)



- The evolution of an ensemble of initial conditions in physical space reveals coherent structures;
- We iterate an ensemble of initial conditions *n* times and then verify how close each of the *nth* iterated initial condition of the ensemble is from a periodic attractor.
- We determine that the *nth* iterated point is in the neighborhood of a periodic orbit, we associate to this point a positive real number.
- Regions with the same hue indicate which initial points will be after *n* iterations in the neighborhood of the same periodic attractor, while the saturation of each point in the region indicates how close its *nth* iteration will be from the periodic attractor.

Nontrivial Time Scales

$$x_{k+1} = (x_k + y_k) \bmod 2\pi + \delta$$
$$y_{k+1} = (1 - v)y_k + f_0 \sin(x_k + y_k) + \delta$$
$$\delta = small\ amplitude\ noise$$



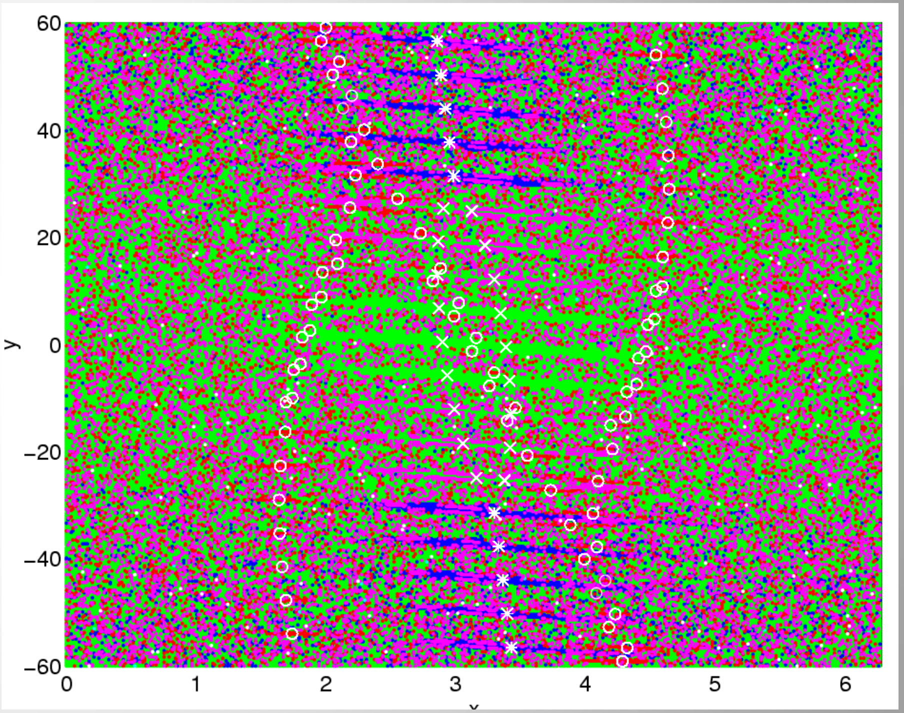
- This complexly interwoven interconnection between the coherent or periodic and random or chaotic structures reflects the appearance of nontrivial time scales in the system.
- We have the mean escape times $<T_i>$ for some of the attractors and the average length of the chaotic transient $<\tau>$ associated with the random structure for different values of noise amplitude;
- The mean escape time is, in general, different for different attractors, for the same noise amplitude;
- The average length of the chaotic transient $<\tau>$ is related to the dimension and the Lyapunov exponents of the chaotic saddles that are embedded in the fractal basin boundary.

Single Rotor with small dissipation and noise:

- Its behavior is neither completely ordered and predictable nor completely random and unpredictable;
- its time evolution reveals patterns and structures over various time and spatial scales;
- This pattern forms hierarchies, *i.e.*, nontrivial structures over a wide range of scales;
- The interconnection among the structures is complicated;
- $\Rightarrow$ the single rotor with noise *can be characterized as a complex system*, regardless for the fact that is a system of low (just two!) dimension.
- The same conclusion follows when similar arguments are applied to other families of multistable systems.

Controlling Complexity 1/2

- for a complex system the unstable chaotic sets in the basin boundaries provide the necessary sensitivity and flexibility to drive the system dynamics toward a specific "ordered" behavior, using *small perturbations*.
- "ordered" ≡ stabilization of one of the metastable attracting sets of the system.

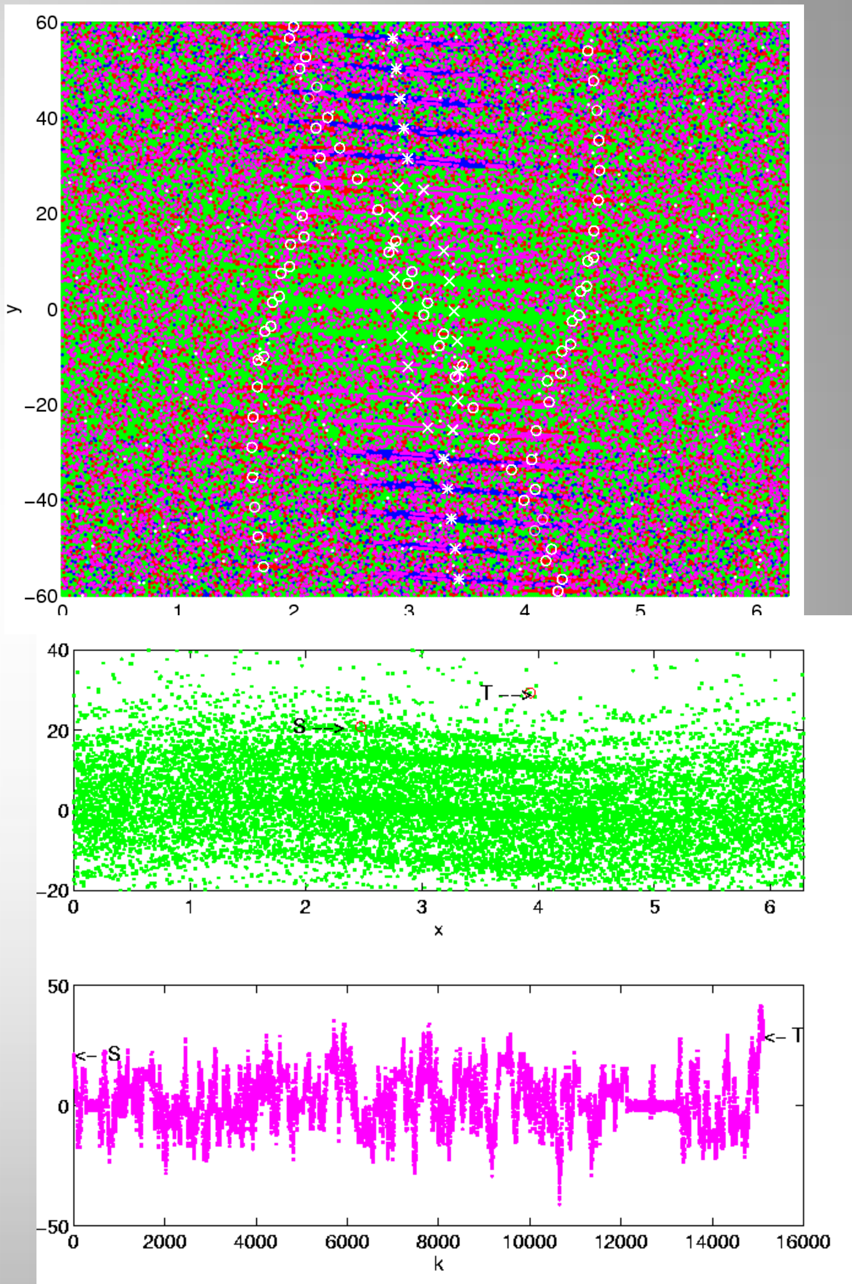


Controlling Complexity 1/2

- “ordered” $\equiv$ trajectory evolving in the neighborhood of fixed (periodic) point x*;
- $x_{n+1} = F(x_n) + \delta = \widetilde{F}(x_n)$

1. The system is left evolving by itself, until it comes close to the desired “ordered” behavior;
2. Linearize the system in the neighborhood of x*:
   $$F(x^* + \varepsilon) \sim x^* + DF(x^*) \times \varepsilon + \delta$$
3. The trajectory can be stabilized by the addition of a controlling term
   $$-DF(x^*) \times (x_i - x^*):$$
   $$\hat{x}_{i+1} = F(x_i) + \delta - DF(x^*) \times (x_i - x^*);$$

Problem: *Transport time* can be excessively long!



How to reduce the transport time?

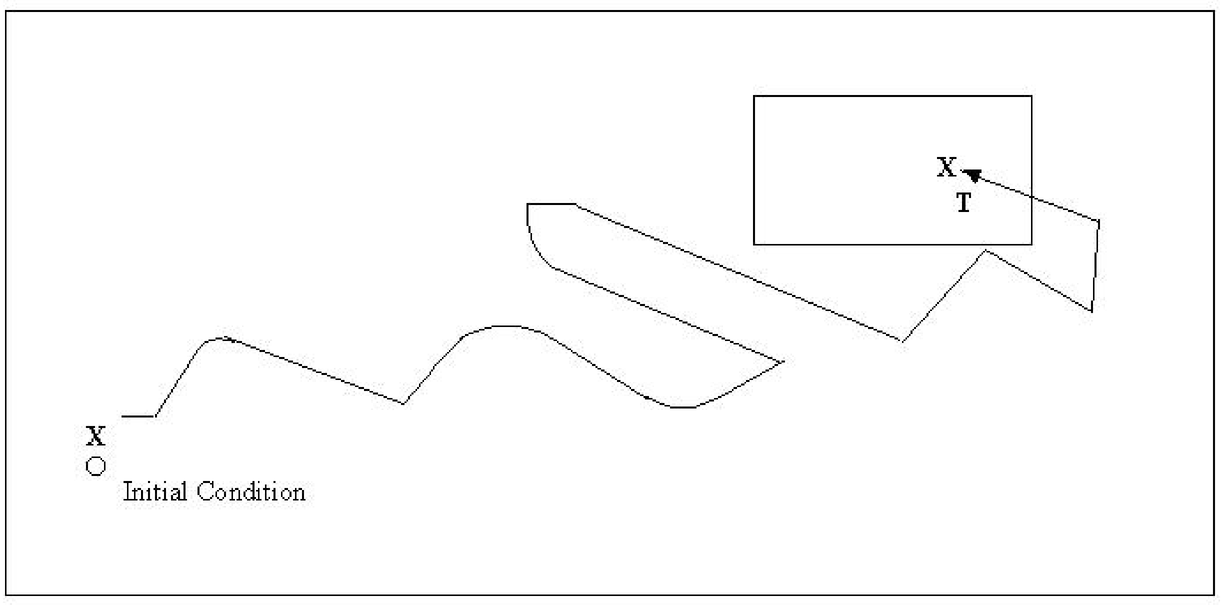
☑ *By guiding trajectories*!

☑ Combine targeting type of control problem fo chaotic systems with techniques used in system control theory:

- Regions of “random” behavior (chaotic transient) <= use “targeting” type of control for chaotic systems;
- Regions of “ordered” behavior <= use traditional system control theory methods.

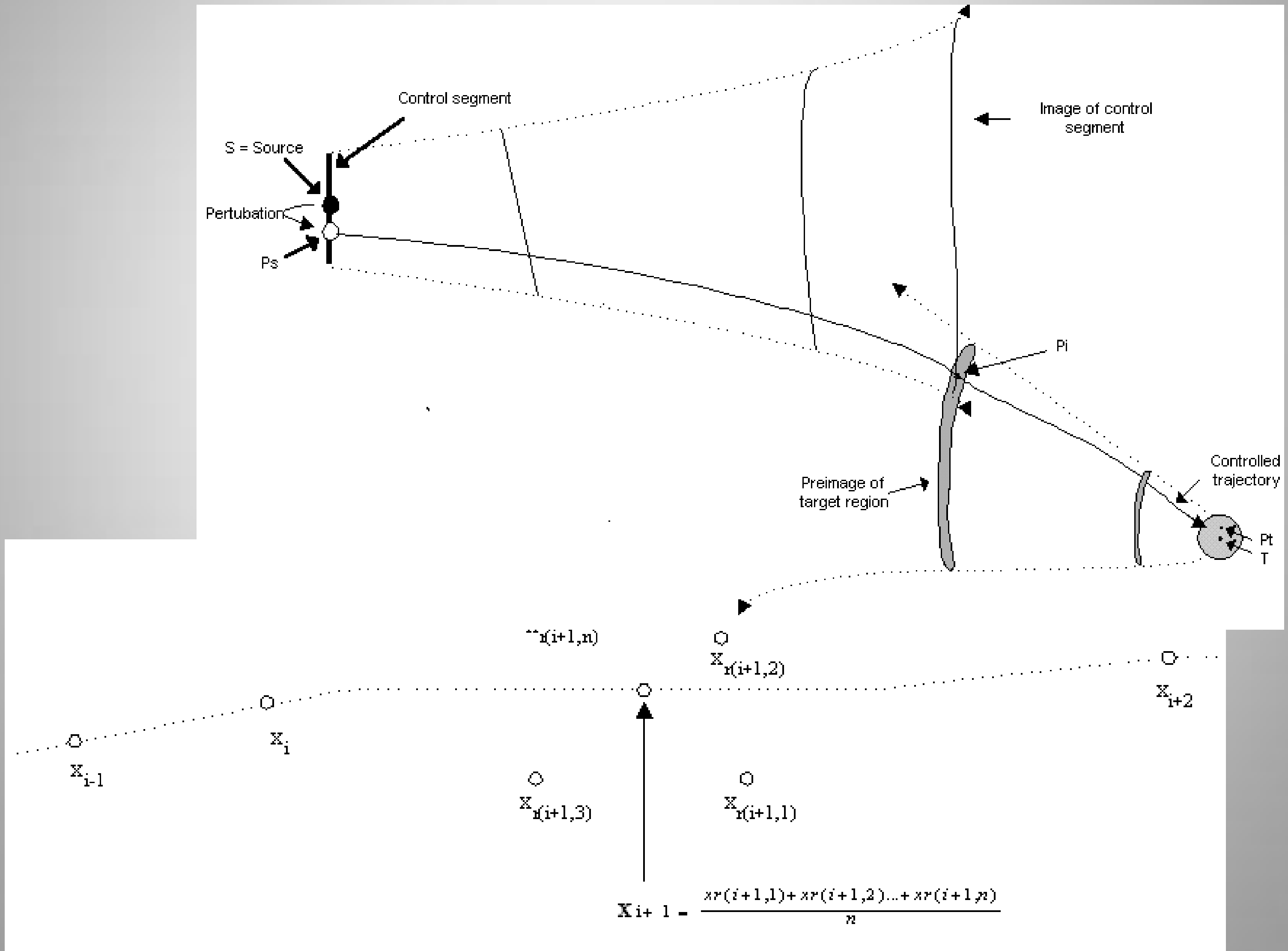
Targeting type of Control of Chaos

- Presence of chaos
- Inherent exponential sensitivity
- Targeting: a procedure to quickly direct a trajectory from o to a small region around T by using small perturbations to some available parameter.



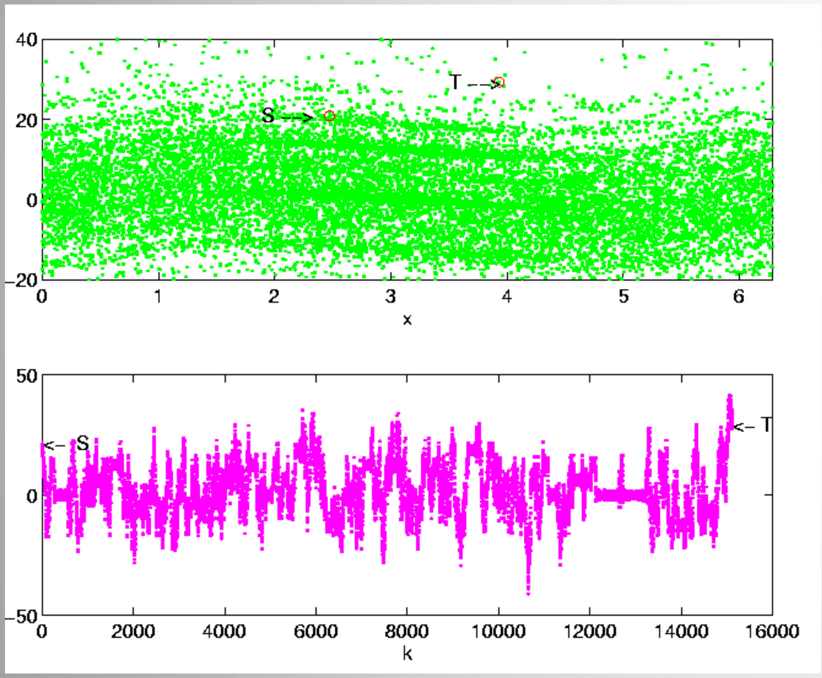
Guiding trajectories in regions of "random" behavior:

- Modified forward-backward targeting method.

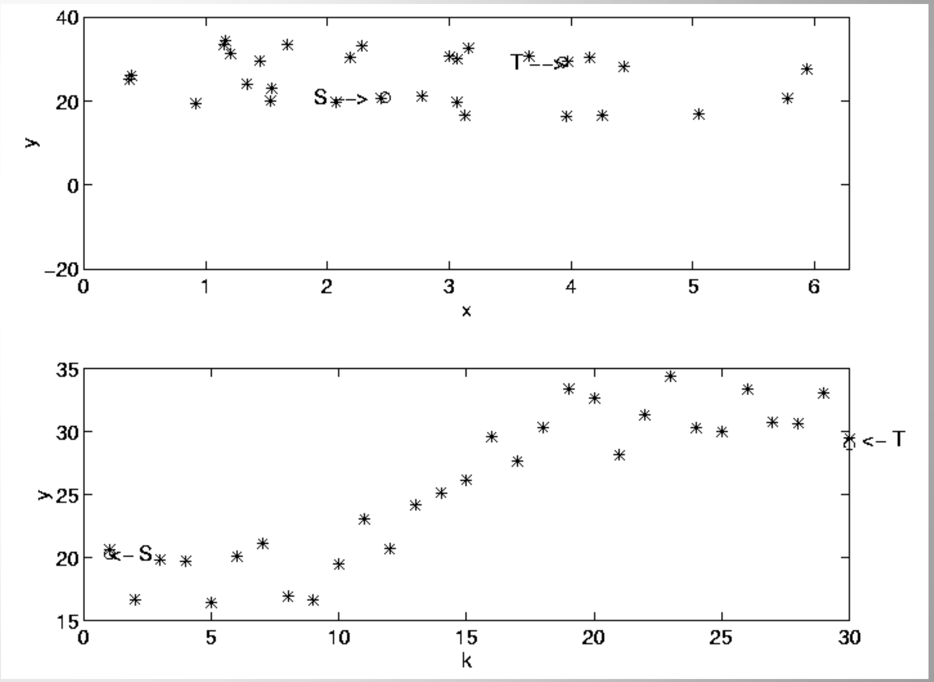


Effect of the Targeting Procedure

- Without targeting



- With targeting



Guiding Trajectories in regions of “ordered” behavior 1/2

- Linearize the system in the neighborhood of x*

$$x_{k+1} = Ax_k\,,$$

where $A = DF(x^*)$;

- Introduce an input term

$$x_{k+1} = Ax_k + Bu_k\,,$$

where u_k is a vector of inputs & *B* is a constant matrix (states how the inputs influence the state);

- Goal: pick u_k to minimize the *cost function*

$$J = \frac{1}{2}\sum_{k=0}^{N}(x_k^t Q_1 x_k + u_k^t Q_2 u_k).$$

Guiding Trajectories in regions of “ordered” behavior 1/2

$$J = \frac{1}{2}\sum_{k=0}^{N}(x_k^t Q_1 x_k + u_k^t Q_2 u_k).$$

- Minimizing by using Lagrange multipliers...

$$u_k = -K_k x_k,$$

with:

$$M_{k+1} + S_{k+1} - S_{k+1}B(Q_2 + B^t S_{k+1}B)^{-1}B^t S_{k+1}$$

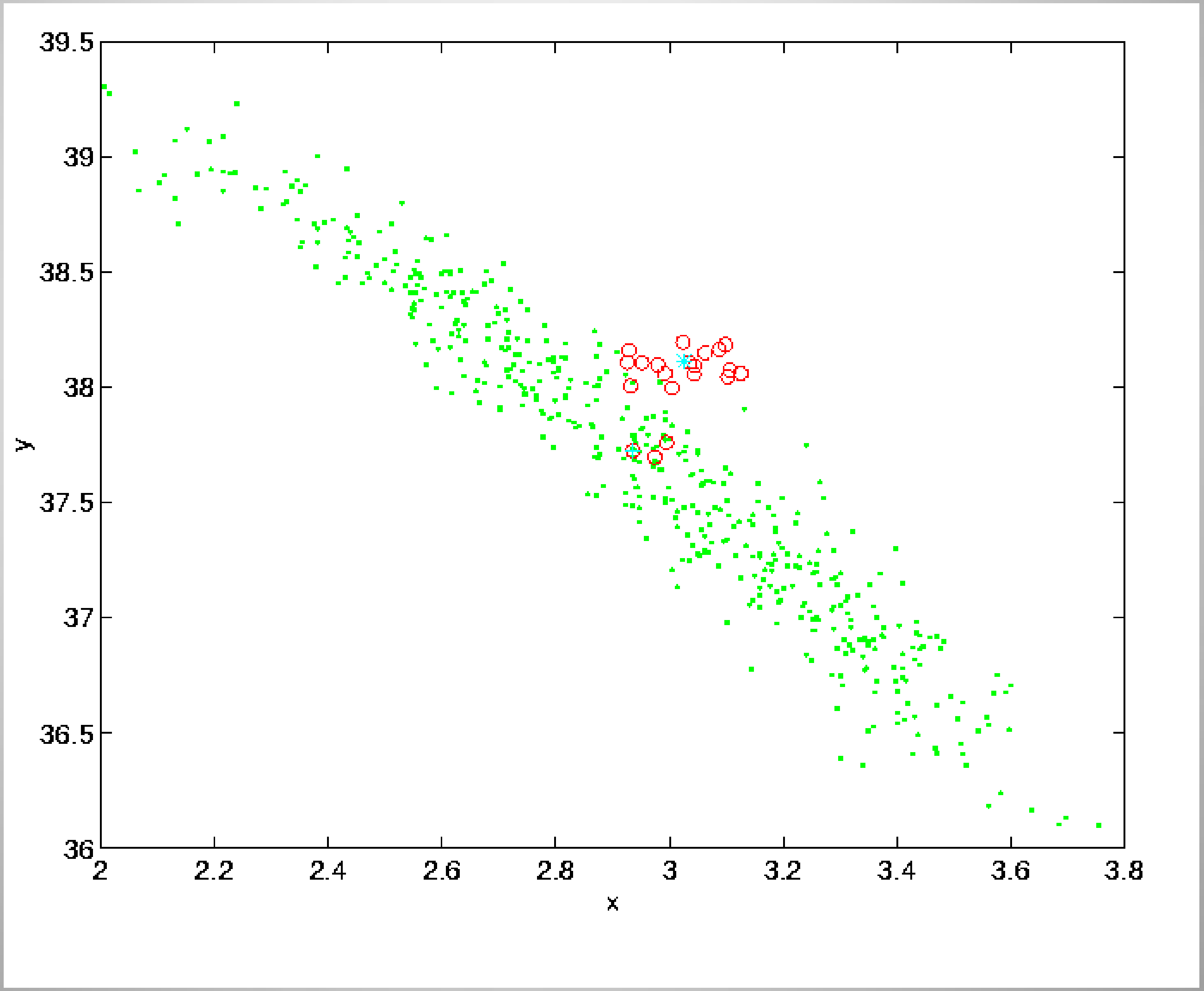
$$S_k = A^t M_{k+1}A + Q_1$$

$$K_k = (Q_2 + B^t S_{k+1}B)^{-1}B^t S_{k+1}A$$

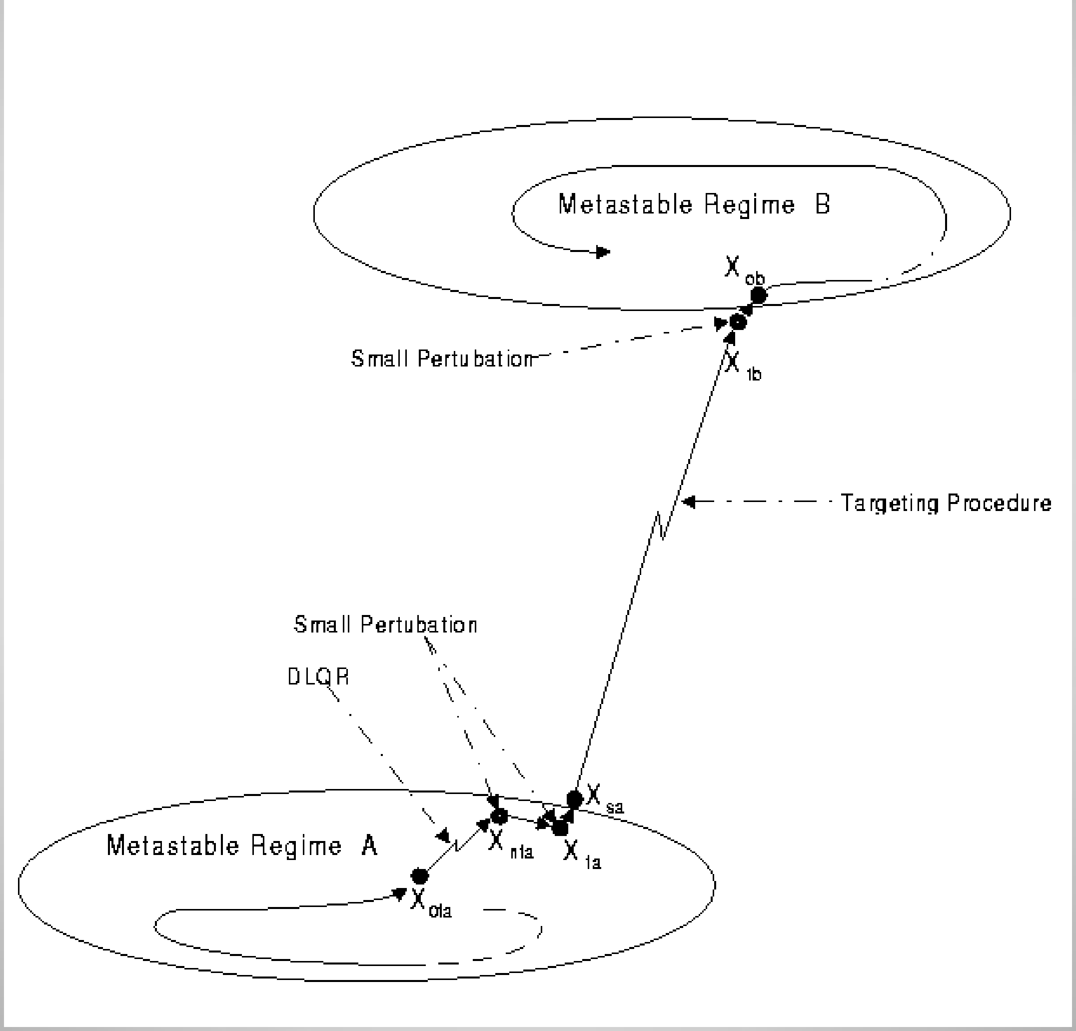
- The equation must be solved backward, with the conditions (*two point boundary-value problem*):

$S_N = Q_1$ and $K_N = 0.$

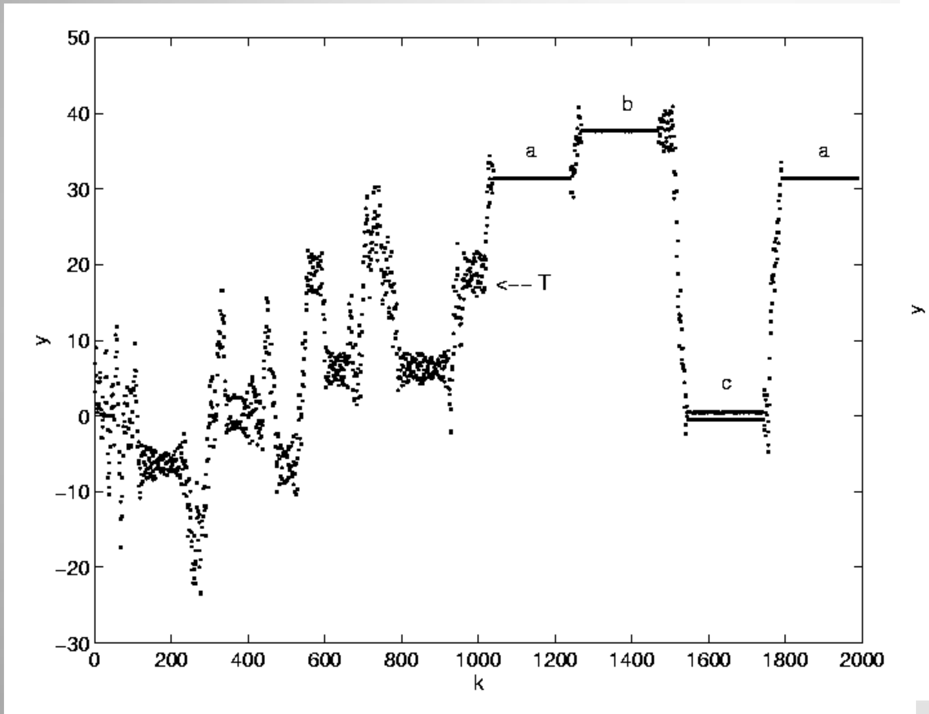
Discrete Linear Quadratic Regulator Action

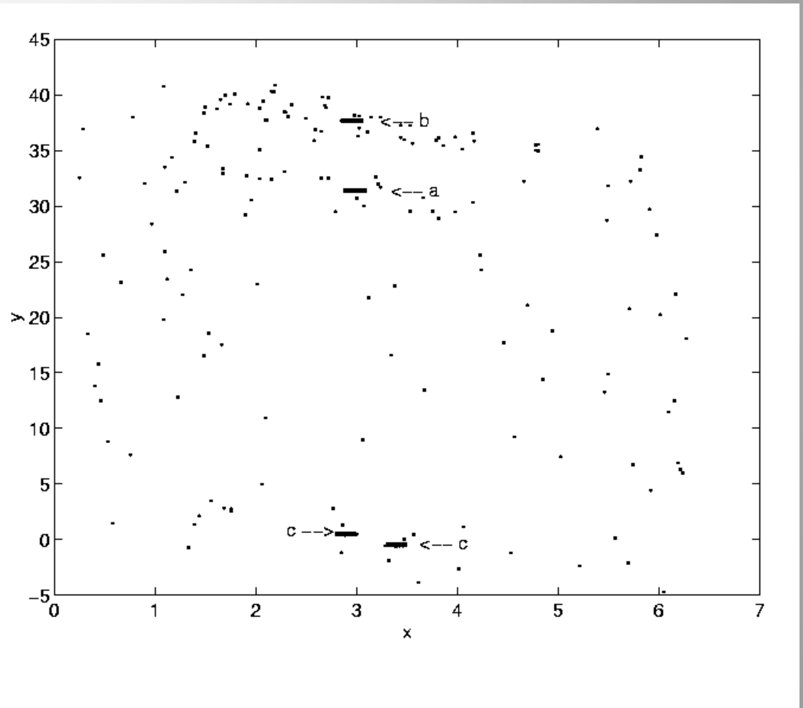


Combining the Previous Procedures



Controlling the Complexity
Changing the Evolution of the System





Conclusion:

- Targeting type of control for chaotic systems can be successfully applied to guide trajectories located in regions of “random” behavior in Complex Systems;
- Traditional system control theory methods can be successfully used to guide trajectories located in regions of “ordered” behavior in Complex Systems;
- The combination of both approaches results in an efficient control strategy to manipulate complex system dynamics.
- That combination can be considered not only for the case of Complex Systems, but also for any system in which complicated dynamics occur.

THANKS

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