COMPRESSIVE SENSING IN NON-LINEAR DYNAMICS

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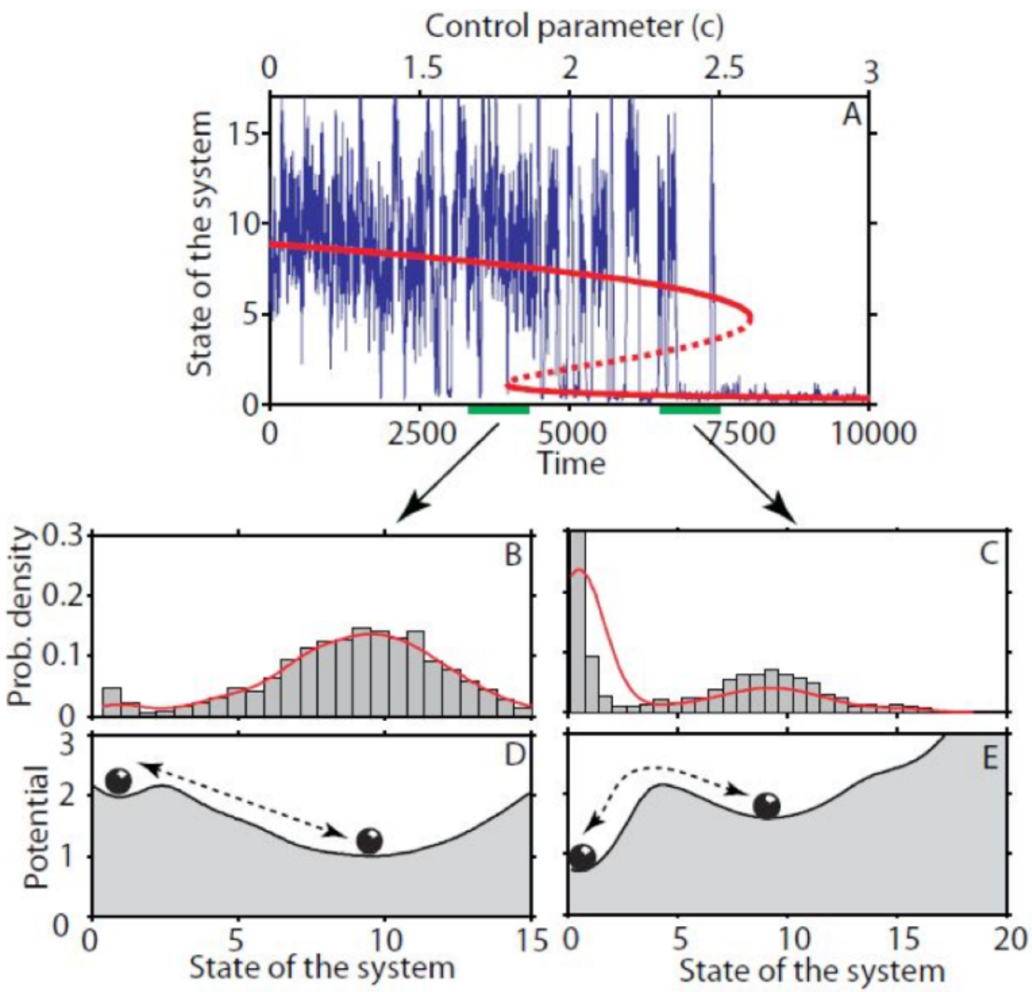
International Workshop on Mathematics of Climate Change and Natural Disasters

Motivations

1) Is it possible to **predict a catastrophe**?



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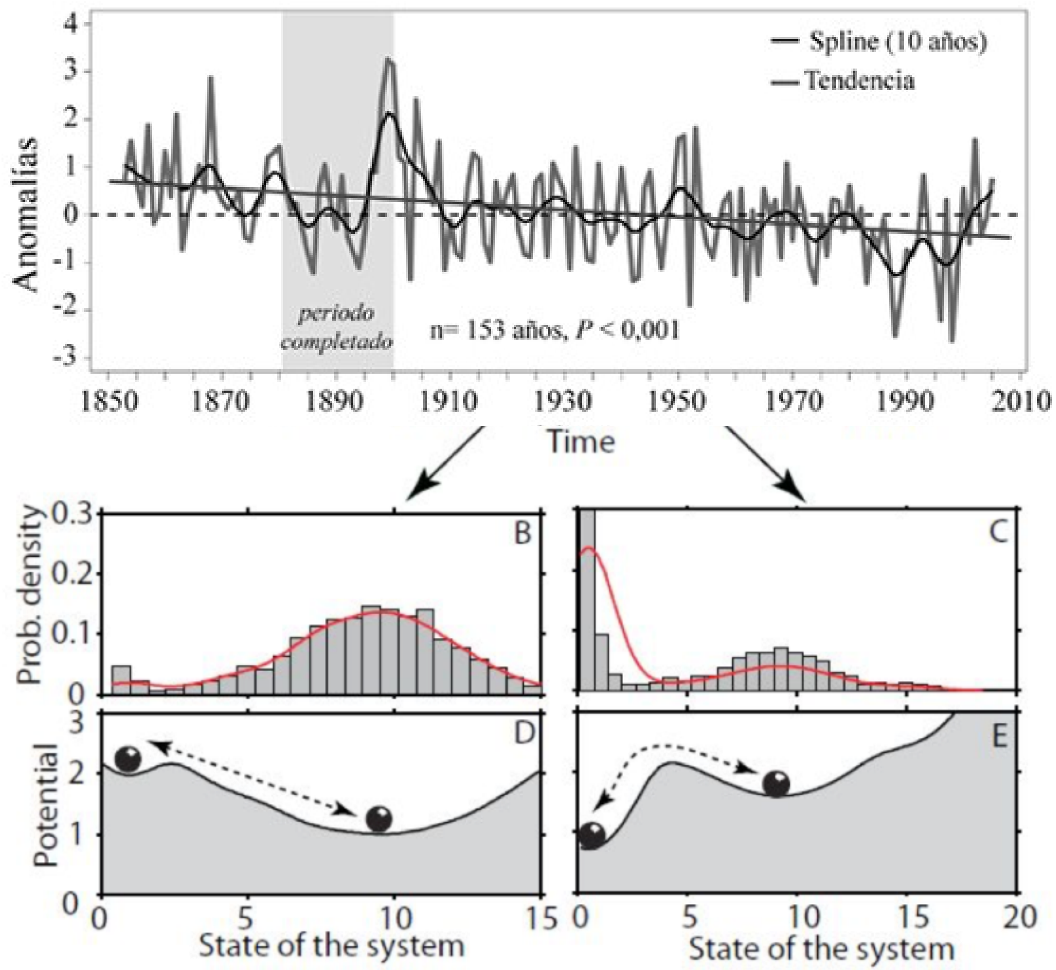
1) Is it possible to predict a catastrophe?

   A: design of an **early-warning indicator**

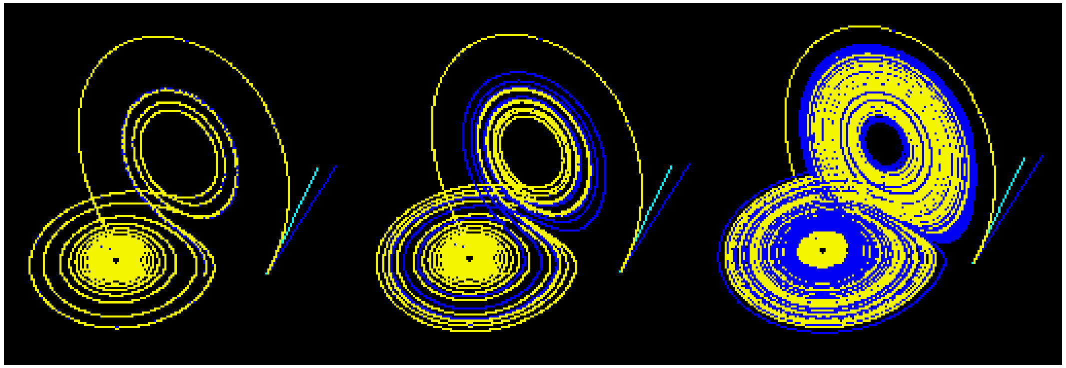


Motivations

1) Is it possible to predict a catastrophe?

   A: design of **early-warning indicators**

2) Is it possible to **predict chaotic trajectories from minimal data**?



$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = x(\rho - z) - y \\ \dot{z} = x\,y - \beta z \end{cases}$$

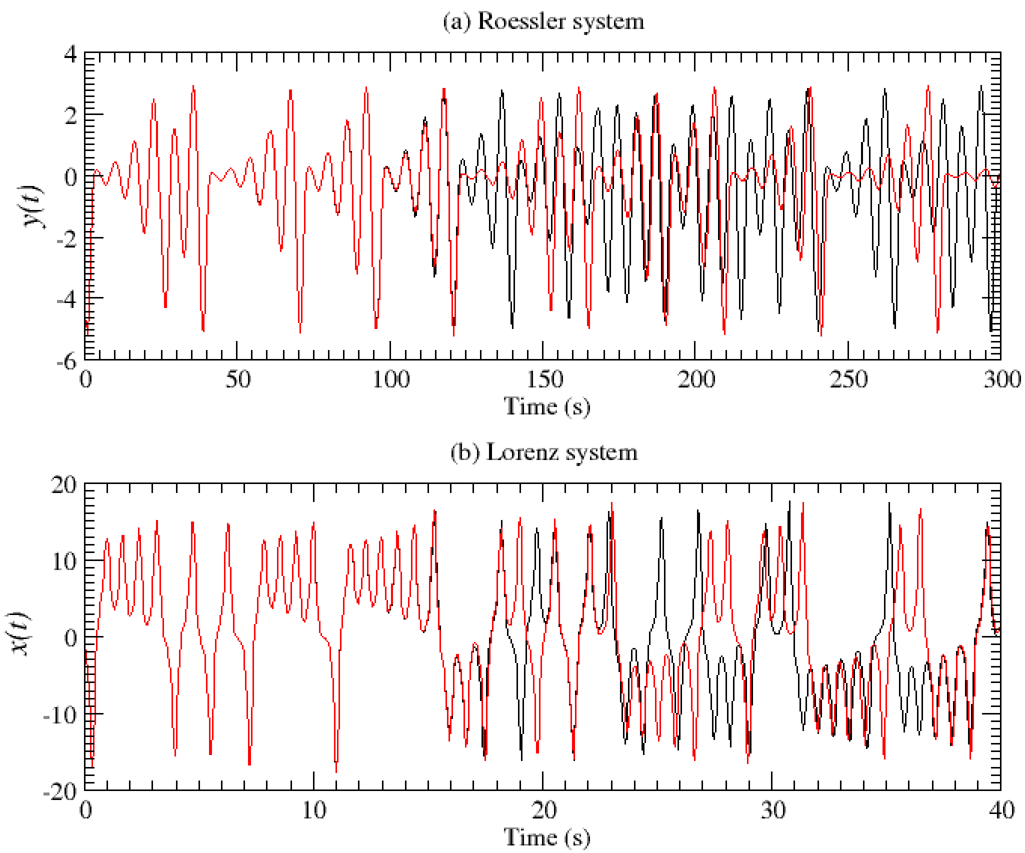
Motivations

1) Is it possible to predict a catastrophe?

   A: design of **early-warning indicators**

2) Is it possible to predict chaotic trajectories from minimal data?

   A: **increasing predictability** without using Nyquist-Shannon limit or Takens theorem



Compressive Sensing (CS), also known as Sparse Modelling, approach

Modelling goal

$\vec{F}_\mu\,?$

- Autonomus dynamical system:

$$\dot{\vec{x}} = \vec{F}_\mu(\vec{x}),\quad \vec{x}\in\mathbb{R}^D,\ \mu\in\mathbb{R}^P\quad \vec{F}: U\subset\mathbb{R}^D\rightarrow\mathbb{R}^D$$

- Non-autonomus dynamical system:

$$\dot{\vec{x}} = \vec{F}_\mu(t,\vec{x}),\quad \vec{x}\in\mathbb{R}^D,\ \mu\in\mathbb{R}^P\quad \vec{F}: I\subset\mathbb{R}\times U\subset\mathbb{R}^D\rightarrow\mathbb{R}^D$$

- Coupled autonomous dynamical systems:

$$\dot{\vec{x}}_i = \vec{f}_i(\vec{x}_i) + \sigma\sum_{j=1}^{N} A_{ij}\left[\vec{h}(\vec{x}_i) - \vec{h}(\vec{x}_j)\right] = \vec{F}_{i,\mu}(\vec{X}),\quad i=1,\ldots,N$$

- **Set of observations:** $\{\vec{x}(t_i)\}_{i=1}^{T}$ **, hence** $\{\dot{\vec{x}}(t_i)\}_{i=1}^{T}$

Observations ~ time-series recordings

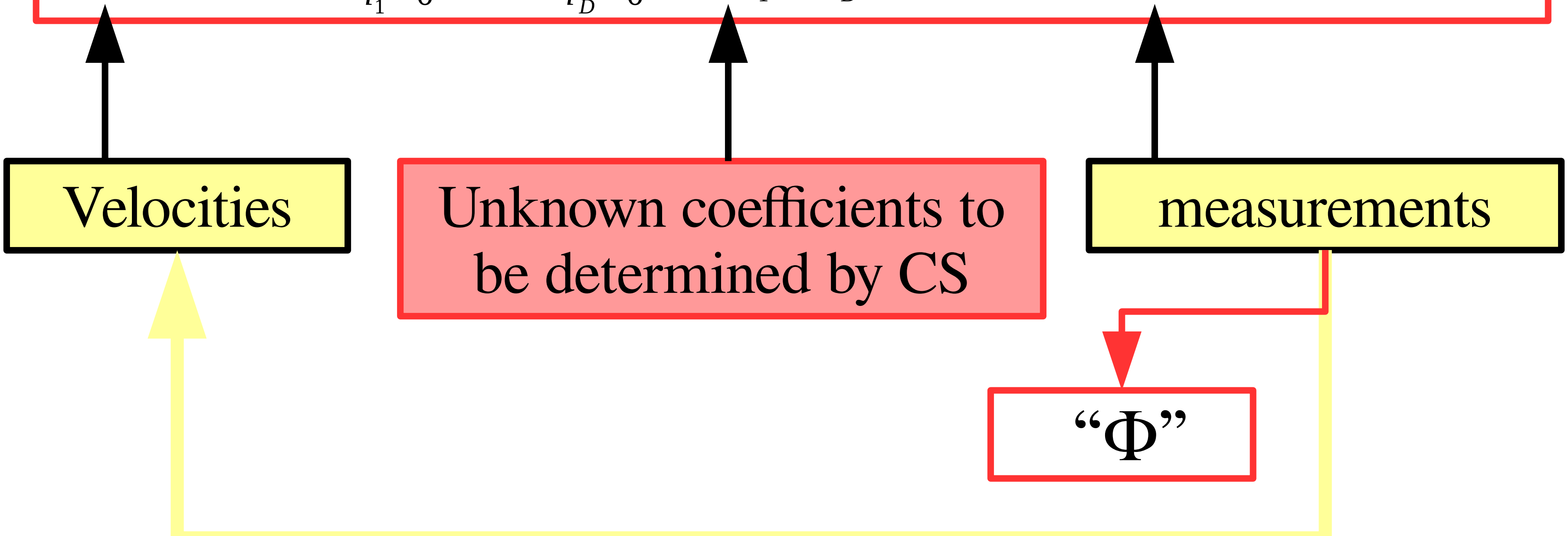
E. Candes, J. Romberg, and T. Tao, *IEEE Trans. Info. Theor.* **52**, 489 (2006)

Framework for CS

- Vectorial field: $\vec{f}(\vec{x}(t)) = \{f_k(\vec{x}(t))\}_{k=1}^{D}$

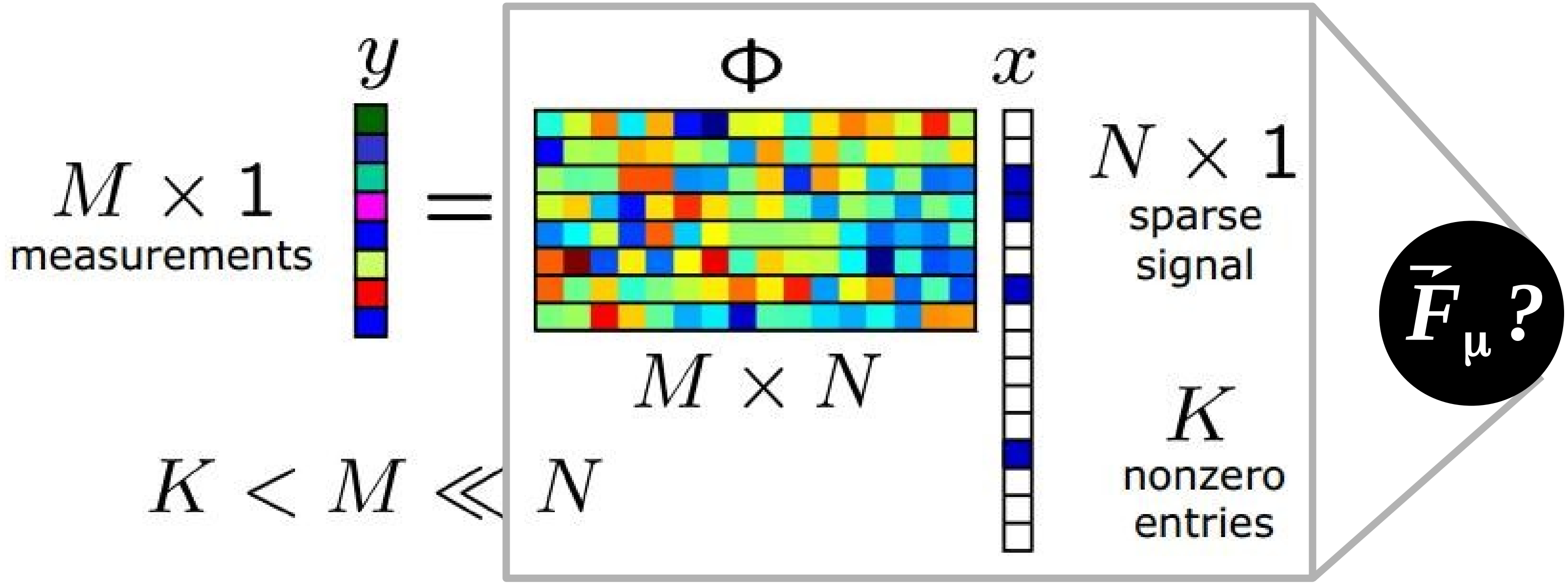
Expansion (orthonormal base) up to order *n*:

$$f_k(\vec{x}(t)) = \sum_{l_1=0}^{n} \dots \sum_{l_D=0}^{n} [a_k]_{l_1,\dots,l_D} x_1(t)^{l_1} \dots x_D(t)^{l_D}, \quad k = 1, \dots, D$$



W.-X. Wang, R. Yang, Y.-C. Lai, V. Kovanis, and C. Grebogi, *Phys. Rev. Lett.* **106**, 154101 (2011)

CS method



- **Problem:** solve an indeterminate linear system by means of minimizing the error using the $L1$-norm and the condition of largest sparsity

$$L1: |\vec{y}|_1 \equiv \sum_{k=1}^{D} |y_k|, \quad L2: |\vec{y}|_2 \equiv \sqrt{\sum_{k=1}^{D} |y_k|^2}, \quad etc.$$

D. Donoho, *IEEE Trans. Info. Theor.* **52**, 1289 (2006)

CS in non-linear dynamics

- $D = 3$ vector flow up to $n = 2 \Rightarrow 27$ coefficients

$$f_x(x(t), y(t), z(t)) \simeq [a_x]_{(0,0,0)} x(t)^0 y(t)^0 z(t)^0 + [a_x]_{(1,0,0)} x(t)^1 y(t)^0 z(t)^0 + \\ + [a_x]_{(0,1,0)} x(t)^0 y(t)^1 z(t)^0 + \ldots + [a_x]_{(2,2,2)} x(t)^2 y(t)^2 z(t)^2$$

$$f_y(x, y, z) \simeq [a_y]_{(0,0,0)} x^0 y^0 z^0 + \ldots, \quad f_z(x, y, z) \simeq [a_z]_{(0,0,0)} x^0 y^0 z^0 + \ldots$$

- Observable measurements (matrix Φ definition):

$$\Phi(t,:) = \{x(t)^0 y(t)^0 z(t)^0, \ldots, x(t)^2 y(t)^2 z(t)^2\} = \vec{\phi}(t), \quad \vec{\phi}(t) \in \mathbb{R}^{1\times 27}$$

- Coefficients (unknowns):

$$\vec{a}_x = \{[a_x]_{(0,0,0)}, [a_x]_{(1,0,0)}, \ldots, [a_x]_{(2,2,2)}\}, \quad \vec{a}_x \in \mathbb{R}^{27\times 1}$$

CS in no-linear dynamics

- $D = 3$ up to order $n = 2 \Rightarrow N = 27$ coefficients

With $M = 5$ measurements at different times:

$$\frac{dx}{dt}(t_1) = f_x(x(t_1), y(t_1), z(t_1)) \simeq \vec{\phi}(t_1) \cdot \vec{a}_x$$

$$\vdots$$

$$\frac{dx}{dt}(t_5) = f_x(x(t_5), y(t_5), z(t_5)) \simeq \vec{\phi}(t_5) \cdot \vec{a}_x$$

$$\Rightarrow \quad (d\vec{X})_{5\times1} \simeq \mathbf{\Phi}_{5\times27} (\vec{a}_x)_{27\times1}$$

Analogously:

$$(d\vec{Y})_{5\times1} \simeq \mathbf{\Phi}_{5\times27} (\vec{a}_y)_{27\times1}$$

$$(d\vec{Z})_{5\times1} \simeq \mathbf{\Phi}_{5\times27} (\vec{a}_z)_{27\times1}$$

- The unknown coefficients are usually sparse!!

W.-X. Wang, R. Yang, Y.-C. Lai, V. Kovanis, and C. Grebogi, *Phys. Rev. Lett.* **106**, 154101 (2011)

CS Results

1) Vectorial field modelling
    1) DS: Parameter inference
    2) CDS: Network and parameter inference
2) Bifurcation analysis
    1) Cascade prediction: crisis
    2) "Early warnings" (for non-autonomous DS)

$$\|\vec{y}-\vec{y}_N\|_2 = K_{2,p}\|\vec{y}\|_p (N+1)^{1/2-1/p}$$

$$M = O(n\log(D))$$

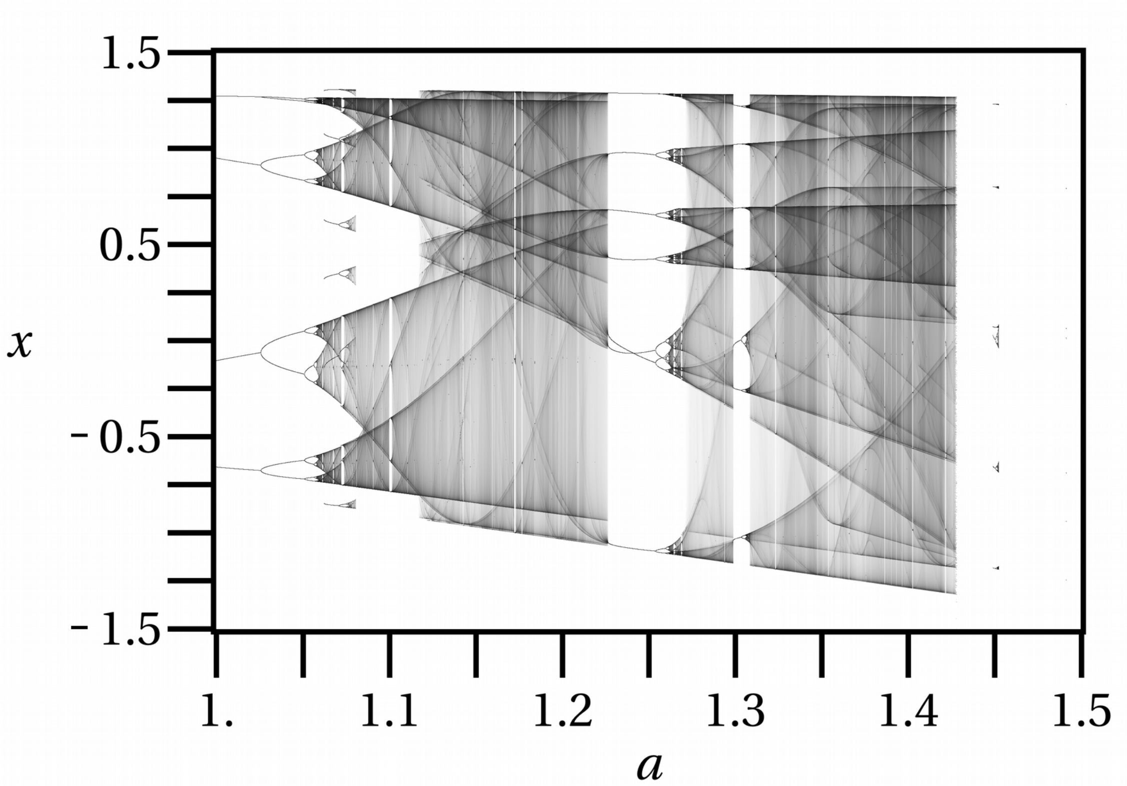
N, being the number of important coefficients ~ n

E. Candes, J. Romberg, and T. Tao, *IEEE Trans. Info. Theor.* **52**, 489 (2006)

Results: parameters

- Example: Henón maps

$$\begin{cases} x_{n+1} = 1 - a x_n^2 + y_n \\ y_{n+1} = b x_n \end{cases}$$

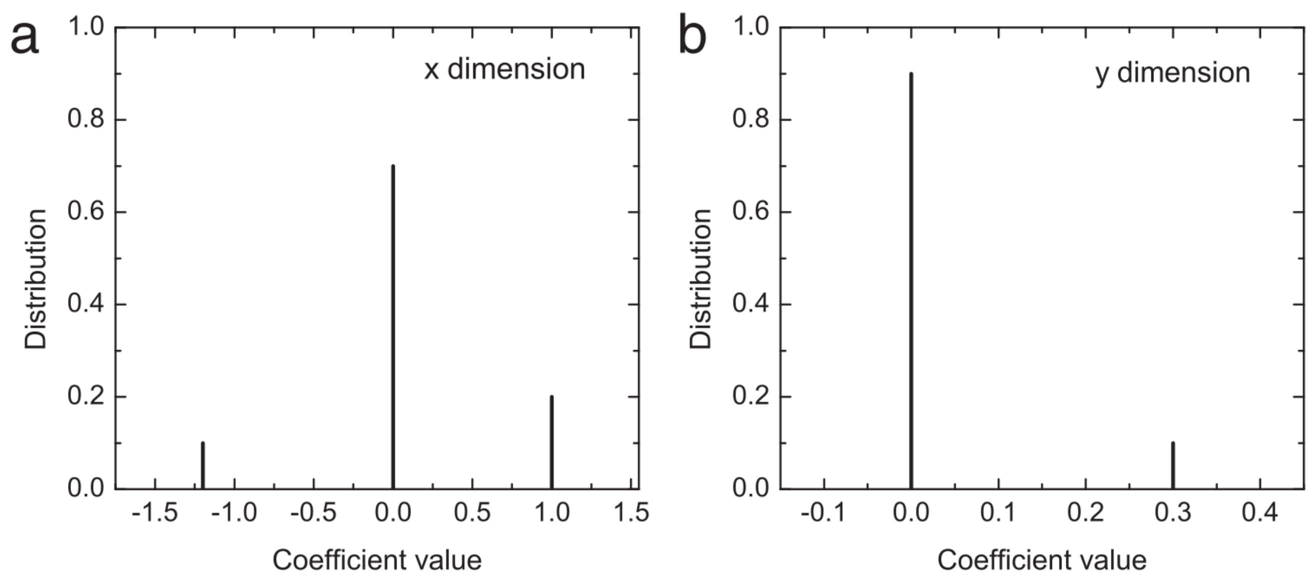


$a = 1.4, \quad b = 0.3$

Results: parameters

- Henón map, with trajectories using $a = 1.2, \;\; b = 0.3$
- Applying CS for $D = 2$, order $n = 2$, and $M = 8$:

$N = 16$ unknown coeff., i.e., $\{1, x, y, xy, x^2, y^2, x^2 y, y^2 x, x^2 y^2\}$



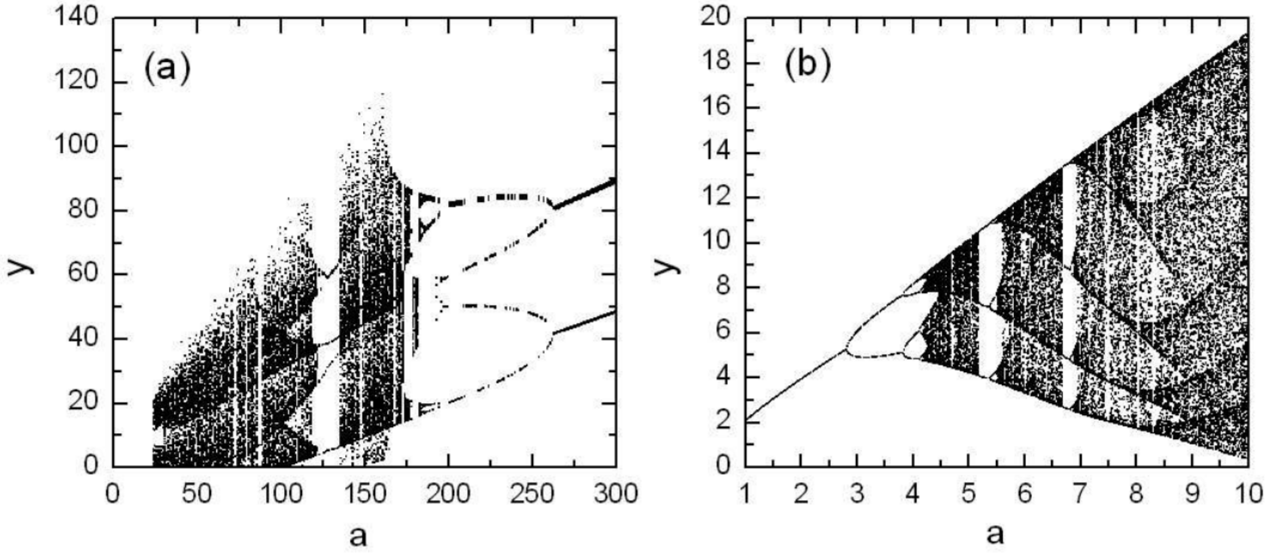
W.-X. Wang, Y.-C. Lai, and C. Grebogi, *Phys. Rep.* **644**, 1-76 (2016)

Resultados: crisis

- Example: Lorenz and Rössler attractors

$N = 35$

$M = 18$



$$\begin{cases} \dot{x} = 10(y - x) \\ \dot{y} = x(a - z) - y \\ \dot{z} = xy - (2/3)z \end{cases}$$

$$\begin{cases} \dot{x} = -y - z \\ \dot{y} = x + (2/10)y \\ \dot{z} = (2/10) + z(x - a) \end{cases}$$

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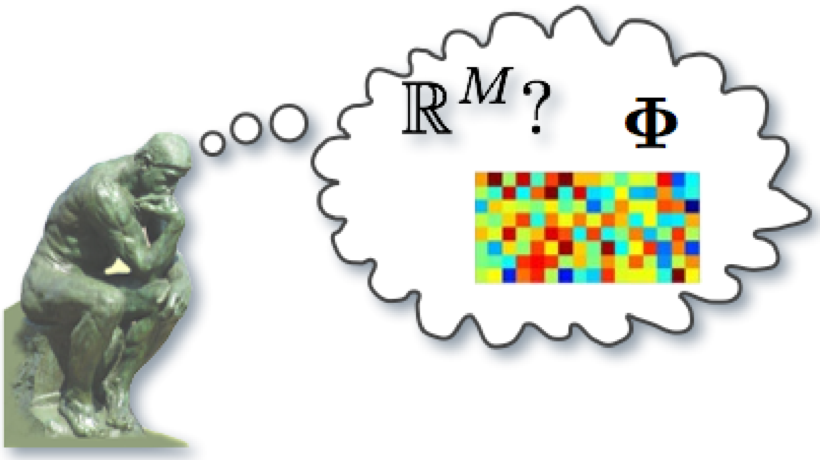
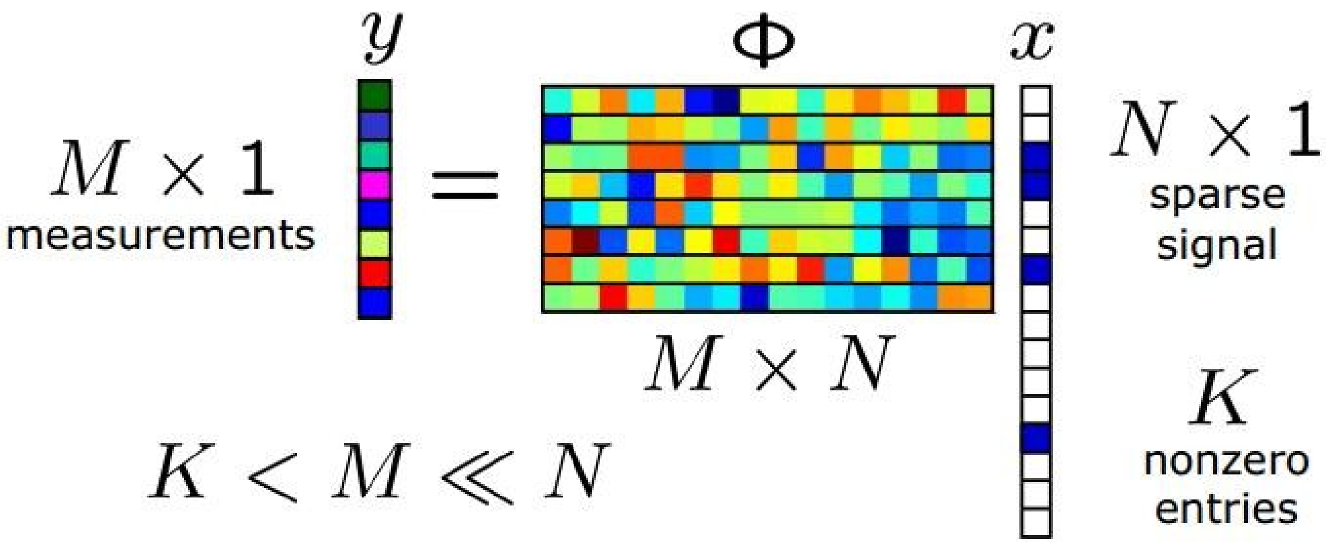
CS method scope

- Which is the correct "Φ"?

A: problem-dependent





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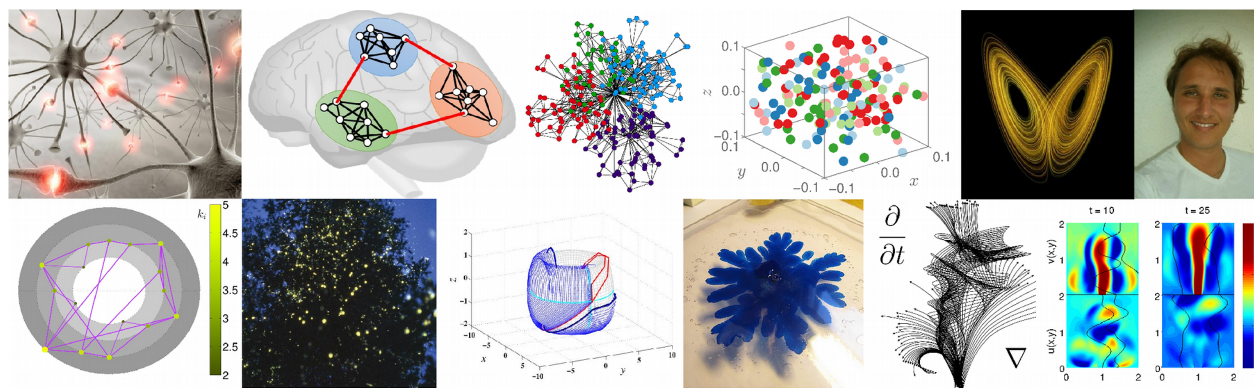
THANKS!

MEFNL group web

Personal web



ICSMB group web