Lecture 2

Hyperbolic AMROC solvers

Course *Block-structured Adaptive Mesh Refinement in C++*

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Outline

- Conservation laws
  - Basics of finite volume methods
  - Splitting methods, second derivatives

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Hyperbolic Conservation Laws

$$\frac{\partial}{\partial t}\mathbf{q}(\mathbf{x},t) + \sum_{n=1}^{d} \frac{\partial}{\partial x_n}\mathbf{f}_n(\mathbf{q}(\mathbf{x},t)) = 0\,, \quad D \subset \{(\mathbf{x},t) \in \mathbb{R}^d \times \mathbb{R}_0^+\}$$

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Definition (Hyperbolicity)

$\mathbf{A}(\mathbf{q},\nu) = \nu_1\mathbf{A}_1(\mathbf{q}) + \cdots + \nu_d\mathbf{A}_d(\mathbf{q})$ with $\mathbf{A}_n(\mathbf{q}) = \partial\mathbf{f}_n(\mathbf{q})/\partial\mathbf{q}$ has M real eigenvalues $\lambda_1(\mathbf{q},\nu) \le ... \le \lambda_M(\mathbf{q},\nu)$ and M linear independent right eigenvectors $\mathbf{r}_m(\mathbf{q},\nu)$.

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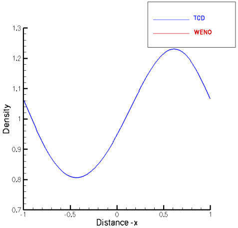
If $\mathbf{f}_n(\mathbf{q})$ is nonlinear, classical solutions $\mathbf{q}(\mathbf{x},t) \in \mathrm{C}^1(D,S)$ do not generally exist, not even for $\mathbf{q}_0(\mathbf{x}) \in \mathrm{C}^1(\mathbb{R}^d,S)$ [Majda, 1984], [Godlewski and Raviart, 1996], [Kröner, 1997]



Example: Euler equations

Weak solutions

Integral form (Gauss's theorem):

$$\int_{\Omega} \mathbf{q}(\mathbf{x}, t+\Delta t)\, d\mathbf{x} - \int_{\Omega} \mathbf{q}(\mathbf{x}, t)\, d\mathbf{x}$$

$$+ \sum_{n=1}^{d} \int_{t}^{t+\Delta t} \int_{\partial\Omega} \mathbf{f}_n(\mathbf{q}(\mathbf{o}, t))\, \sigma_n(\mathbf{o})\, d\mathbf{o}\, dt = \int_{t}^{t+\Delta t} \int_{\Omega} \mathbf{s}(\mathbf{q}(\mathbf{x}, t))\, d\mathbf{x}$$

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Theorem (Weak solution)

$\mathbf{q}_0 \in \mathrm{L}^\infty_{loc}(\mathbb{R}^d, S)$. $\mathbf{q} \in \mathrm{L}^\infty_{loc}(D, S)$ *is weak solution if* $\mathbf{q}$ *satisfies*

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$$\int_0^{\infty} \int_{\mathbb{R}^d} \left[\frac{\partial \varphi}{\partial t} \cdot \mathbf{q} + \sum_{n=1}^{d} \frac{\partial \varphi}{\partial x_n} \cdot \mathbf{f}_n(\mathbf{q}) - \varphi \cdot \mathbf{s}(\mathbf{q}) \right] d\mathbf{x}\, dt + \int_{\mathbb{R}^d} \varphi(\mathbf{x}, 0) \cdot \mathbf{q}_0(\mathbf{x})\, d\mathbf{x} = 0$$

for any test function $\varphi \in \mathrm{C}^1_0(D, S)$

Examples

Euler equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_n}(\rho u_n) = 0$$

$$\frac{\partial}{\partial t}(\rho u_k) + \frac{\partial}{\partial x_n}(\rho u_k u_n + \delta_{kn} p) = 0 \; , \quad k = 1, \dots, d$$

$$\frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial x_n}(u_n(\rho E + p)) = 0$$

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have structure

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Examples II

Navier-Stokes equations

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$$\tau_{kn} = \mu\left(\frac{\partial u_n}{\partial x_k} + \frac{\partial u_k}{\partial x_n}\right) - \frac{2}{3}\mu \frac{\partial u_j}{\partial x_j}\delta_{kn}$$

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Type can be either hyperbolic or parabolic

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$I_j = [x_j - \frac{1}{2}\Delta x, x_j + \frac{1}{2}\Delta x[=: [x_{j-1/2}, x_{j+1/2}[$

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Using approximations $\mathbf{Q}_j(t) \approx \frac{1}{|I_j|} \int\limits_{I_j} \mathbf{q}(\mathbf{x}, t)\, dx, \quad \mathbf{s}(\mathbf{Q}_j(t)) \approx \frac{1}{|I_j|} \int\limits_{I_j} \mathbf{s}(\mathbf{q}(\mathbf{x}, t))\, dx$

and numerical fluxes

$\mathbf{F}\left(\mathbf{Q}_j(t), \mathbf{Q}_{j+1}(t)\right) \approx \mathbf{f}(\mathbf{q}(x_{j+1/2}, t)), \quad \mathbf{H}\left(\mathbf{Q}_j(t), \mathbf{Q}_{j+1}(t)\right) \approx \mathbf{h}(\mathbf{q}(x_{j+1/2}, t), \nabla \mathbf{q}(x_{j+1/2}, t))$

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yields after integration (Gauss theorem)

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For instance:

$$\mathbf{Q}_j^{n+1} = \mathbf{Q}_j^n - \frac{\Delta t}{\Delta x}\left[\mathbf{F}\left(\mathbf{Q}_j^n, \mathbf{Q}_{j+1}^n\right) - \mathbf{F}\left(\mathbf{Q}_{j-1}^n, \mathbf{Q}_j^n\right)\right] -$$
$$\frac{\Delta t}{\Delta x}\left[\mathbf{H}\left(\mathbf{Q}_j^n, \mathbf{Q}_{j+1}^n\right) - \mathbf{H}\left(\mathbf{Q}_{j-1}^n, \mathbf{Q}_j^n\right)\right] + \Delta t \mathbf{s}(\mathbf{Q}_j^n)\, dt$$

Splitting methods

Solve homogeneous PDE and ODE successively!

$$\mathcal{H}^{(\Delta t)}: \quad \partial_t \mathbf{q} + \nabla \cdot \mathbf{f}(\mathbf{q}) = 0 \;, \quad \text{IC: } \mathbf{Q}(t_m) \stackrel{\Delta t}{\Longrightarrow} \tilde{\mathbf{Q}}$$

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1st-order dimensional splitting for $\mathcal{H}^{(\cdot)}$:

$$\mathcal{X}_1^{(\Delta t)}: \quad \partial_t \mathbf{q} + \partial_{x_1} \mathbf{f}_1(\mathbf{q}) = 0 \,, \quad \text{IC: } \mathbf{Q}(t_m) \stackrel{\Delta t}{\Longrightarrow} \tilde{\mathbf{Q}}^{1/2}$$
$$\mathcal{X}_2^{(\Delta t)}: \quad \partial_t \mathbf{q} + \partial_{x_2} \mathbf{f}_2(\mathbf{q}) = 0 \,, \quad \text{IC: } \tilde{\mathbf{Q}}^{1/2} \stackrel{\Delta t}{\Longrightarrow} \tilde{\mathbf{Q}}$$

[Toro, 1999]

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or conservatively

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Von Neumann stability analysis: Insert single eigenmode $\hat{Q}(t)e^{ik_1x_1}e^{ik_2x_2}$ into discretization

$$\hat{Q}^{n+1} = \hat{Q}^n + C_1\left(\hat{Q}^n e^{ik_1\Delta x_1} - 2\hat{Q}^n + \hat{Q}^n e^{-ik_1\Delta x_1}\right) + C_2\left(\hat{Q}^n e^{ik_2\Delta x_2} - 2\hat{Q}^n + \hat{Q}^n e^{-ik_2\Delta x_2}\right)$$

with $C_\iota = c\frac{\Delta t}{\Delta x_\iota^2},\ \iota = 1, 2,$

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$$Q_{jk}^{n+1} = Q_{jk}^n + c\frac{\Delta t}{\Delta x_1}\left(H^1_{j+\frac{1}{2},k} - H^1_{j-\frac{1}{2},k}\right) + c\frac{\Delta t}{\Delta x_2}\left(H^2_{j,k+\frac{1}{2}} - H^2_{j,k-\frac{1}{2}}\right)$$

Von Neumann stability analysis: Insert single eigenmode $\hat{Q}(t)e^{ik_1x_1}e^{ik_2x_2}$ into discretization

$$\hat{Q}^{n+1} = \hat{Q}^n + C_1\left(\hat{Q}^n e^{ik_1\Delta x_1} - 2\hat{Q}^n + \hat{Q}^n e^{-ik_1\Delta x_1}\right) + C_2\left(\hat{Q}^n e^{ik_2\Delta x_2} - 2\hat{Q}^n + \hat{Q}^n e^{-ik_2\Delta x_2}\right)$$

with $C_\iota = c\frac{\Delta t}{\Delta x_\iota^2}$, $\iota = 1,2$, which gives after inserting $e^{ik_\iota x_\iota} = \cos(k_\iota x_\iota) + i\sin(k_\iota x_\iota)$

$$\hat{Q}^{n+1} = \hat{Q}^n\left(1 + 2C_1(\cos(k_1\Delta x_1) - 1) + 2C_2(\cos(k_2\Delta x_2) - 1)\right)$$

Conservative scheme for diffusion equation

Consider $\partial_t q - c\Delta q = 0$ with $c \in \mathbb{R}^+$, which is readily discretized as

$$Q_{jk}^{n+1} = Q_{jk}^n + c\frac{\Delta t}{\Delta x_1^2}\left(Q_{j+1,k}^n - 2Q_{jk}^n + Q_{j-1,k}^n\right) + c\frac{\Delta t}{\Delta x_2^2}\left(Q_{j,k+1}^n - 2Q_{jk}^n + Q_{j,k-1}^n\right)$$

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Stability requires

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Stability requires

$$|1 + 2C_1(\cos(k_1\Delta x_1) - 1) + 2C_2(\cos(k_2\Delta x_2) - 1)| \le 1$$

i.e.

$$|1 - 4C_1 - 4C_2| \le 1$$

from which we derive the stability condition

$$0 \le c\left(\frac{\Delta t}{\Delta x_1^2} + \frac{\Delta t}{\Delta x_2^2}\right) \le \frac{1}{2}$$

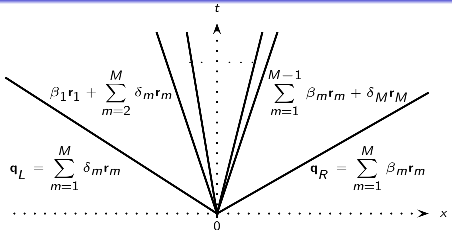
Outline

Linear upwind schemes

Consider Riemann problem

$$\frac{\partial}{\partial t}\mathbf{q}(x,t)+\mathbf{A}\frac{\partial}{\partial x}\mathbf{q}(x,t)=\mathbf{0}\,,\quad x\in\mathbb{R}\,,\quad t>0$$

Linear upwind schemes



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Has exact solution

$$\mathbf{q}(x,t)=\mathbf{q}_L+\sum_{\lambda_m<x/t} a_m\mathbf{r}_m=\mathbf{q}_R-\sum_{\lambda_m\geq x/t} a_m\mathbf{r}_m=\sum_{\lambda_m\geq x/t}\delta_m\mathbf{r}_m+\sum_{\lambda_m<x/t}\beta_m\mathbf{r}_m$$

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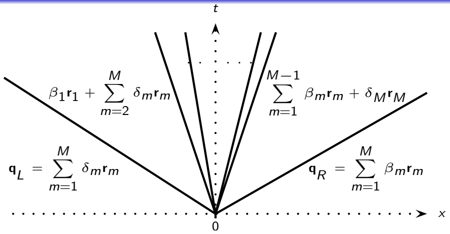

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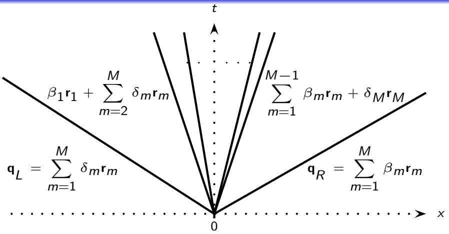

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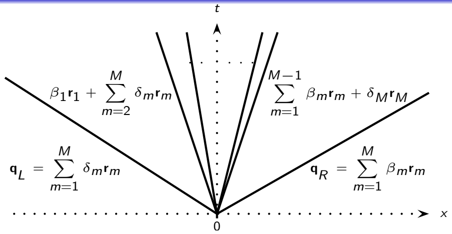
Use $\quad \lambda_m^+=\max(\lambda_m,0)\,,\qquad \lambda_m^-=\min(\lambda_m,0)$

to define $\quad \mathbf{\Lambda}^+:=\mathrm{diag}(\lambda_1^+,\ldots,\lambda_M^+)\,,\qquad \mathbf{\Lambda}^-:=\mathrm{diag}(\lambda_1^-,\ldots,\lambda_M^-)$

Linear upwind schemes



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and $\quad \mathbf{A}^+:=\mathbf{R}\,\mathbf{\Lambda}^+\,\mathbf{R}^{-1}\,,\qquad \mathbf{A}^-:=\mathbf{R}\,\mathbf{\Lambda}^-\,\mathbf{R}^{-1}\qquad$ which gives

$$\mathbf{F}(\mathbf{q}_L,\mathbf{q}_R)=\mathbf{A}\mathbf{q}_L+\mathbf{A}^-\Delta\mathbf{q}=\mathbf{A}\mathbf{q}_R-\mathbf{A}^+\Delta\mathbf{q}=\mathbf{A}^+\mathbf{q}_L+\mathbf{A}^-\mathbf{q}_R$$

with $\Delta\mathbf{q}=\mathbf{q}_R-\mathbf{q}_L$

Flux difference splitting

Godunov-type scheme with $\Delta \mathbf{Q}^n_{j+1/2} = \mathbf{Q}^n_{j+1} - \mathbf{Q}^n_j$

$$\mathbf{Q}^{n+1}_j = \mathbf{Q}^n_j - \frac{\Delta t}{\Delta x}\left(\mathbf{A}^- \Delta \mathbf{Q}^n_{j+1/2} + \mathbf{A}^+ \Delta \mathbf{Q}^n_{j-1/2}\right)$$

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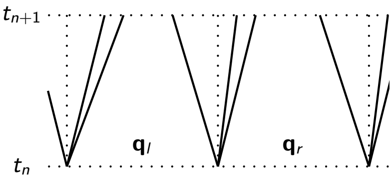
stability condition

$$\max_{j\in\mathbb{Z}} |\hat{\lambda}_{m,j+\frac{1}{2}}| \frac{\Delta t}{\Delta x} \le 1 \;, \quad \text{for all } m = 1, \ldots, M$$

[LeVeque, 1992]

Roe's approximate Riemann solver

Choosing $\hat{\mathbf{A}}(\mathbf{q}_L, \mathbf{q}_R)$ [Roe, 1981]:



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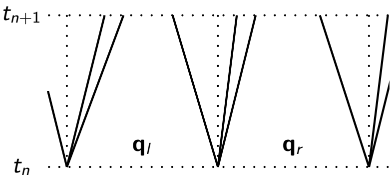

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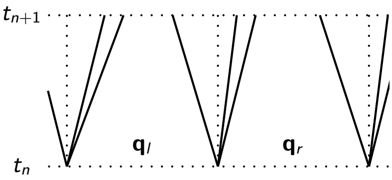

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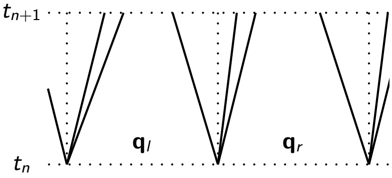

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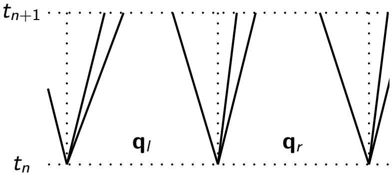

For Euler equations:

$$\hat{\rho} = \frac{\sqrt{\rho_L}\rho_R + \sqrt{\rho_R}\rho_L}{\sqrt{\rho_L} + \sqrt{\rho_R}} = \sqrt{\rho_L \rho_R} \quad \text{and} \quad \hat{v} = \frac{\sqrt{\rho_L} v_L + \sqrt{\rho_R} v_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \quad \text{for } v = u_n, H$$

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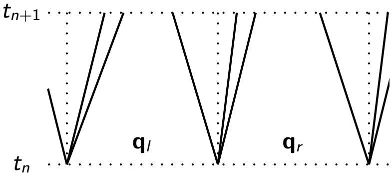

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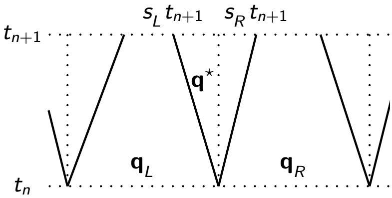
Wave decomposition: $\Delta\mathbf{q} = \mathbf{q}_r - \mathbf{q}_l = \sum_m a_m \hat{\mathbf{r}}_m$

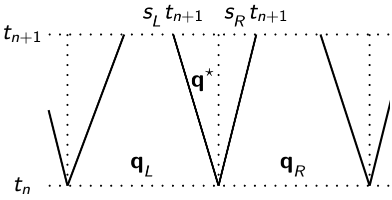
$$\begin{aligned} \mathbf{F}(\mathbf{q}_L, \mathbf{q}_R) &= \mathbf{f}(\mathbf{q}_L) + \sum_{\hat{\lambda}_m < 0} \hat{\lambda}_m \, a_m \, \hat{\mathbf{r}}_m = \mathbf{f}(\mathbf{q}_R) - \sum_{\hat{\lambda}_m \geq 0} \hat{\lambda}_m \, a_m \, \hat{\mathbf{r}}_m \\ &= \frac{1}{2}\left(\mathbf{f}(\mathbf{q}_L) + \mathbf{f}(\mathbf{q}_R) - \sum_m |\hat{\lambda}_m| \, a_m \, \hat{\mathbf{r}}_m \right) \end{aligned}$$

Harten-Lax-Van Leer (HLL) approximate Riemann solver



$$\bar{\mathbf{q}}(x,t) = \begin{cases} \mathbf{q}_L\,, & x < s_L\, t \\ \mathbf{q}^\star\,, & s_L\, t \le x \le s_R\, t \\ \mathbf{q}_R\,, & x > s_R\, t \end{cases}$$

Harten-Lax-Van Leer (HLL) approximate Riemann solver



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$$\mathbf{F}_{HLL}(\mathbf{q}_L, \mathbf{q}_R) = \begin{cases} \mathbf{f}(\mathbf{q}_L)\,, & 0 < s_L\,, \\ \dfrac{s_R \mathbf{f}(\mathbf{q}_L) - s_L \mathbf{f}(\mathbf{q}_R) + s_L s_R (\mathbf{q}_R - \mathbf{q}_L)}{s_R - s_L}\,, & s_L \le 0 \le s_R\,, \\ \mathbf{f}(\mathbf{q}_R)\,, & 0 > s_R\,, \end{cases}$$

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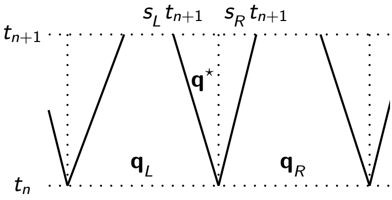

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Euler equations:

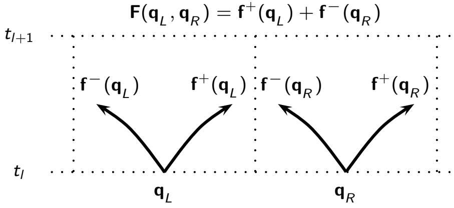
$$s_L = \min(u_{1,L} - c_L, u_{1,R} - c_R) \,, \quad s_R = \max(u_{1,L} + c_l, u_{1,R} + c_R)$$

[Toro, 1999], HLLC: [Toro et al., 1994]

Flux vector splitting

Splitting

$$\mathbf{f}(\mathbf{q}) = \mathbf{f}^{+}(\mathbf{q}) + \mathbf{f}^{-}(\mathbf{q})$$



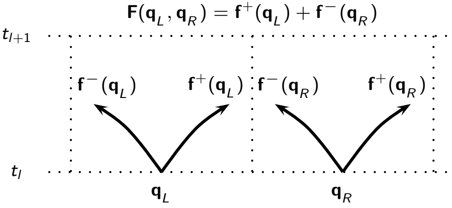
Flux vector splitting

Splitting

$$\mathbf{f}(\mathbf{q}) = \mathbf{f}^+(\mathbf{q}) + \mathbf{f}^-(\mathbf{q})$$

derived under restriction $\hat{\lambda}_m^+ \geq 0$ and $\hat{\lambda}_m^- \leq 0$ for all $m = 1, \ldots, M$ for

$$\hat{\mathbf{A}}^+(\mathbf{q}) = \frac{\partial \mathbf{f}^+(\mathbf{q})}{\partial \mathbf{q}}, \quad \hat{\mathbf{A}}^-(\mathbf{q}) = \frac{\partial \mathbf{f}^-(\mathbf{q})}{\partial \mathbf{q}}$$



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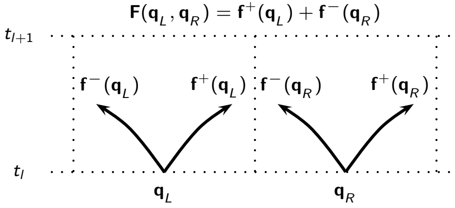
$$\mathbf{f}(\mathbf{q}) = \mathbf{f}^+(\mathbf{q}) + \mathbf{f}^-(\mathbf{q})$$

derived under restriction $\hat{\lambda}_m^+ \ge 0$ and $\hat{\lambda}_m^- \le 0$ for all $m = 1, \ldots, M$ for

$$\hat{\mathbf{A}}^+(\mathbf{q}) = \frac{\partial \mathbf{f}^+(\mathbf{q})}{\partial \mathbf{q}}, \quad \hat{\mathbf{A}}^-(\mathbf{q}) = \frac{\partial \mathbf{f}^-(\mathbf{q})}{\partial \mathbf{q}}$$



plus reproduction of regular upwinding

$$\begin{array}{lllllllll} \mathbf{f}^+(\mathbf{q}) & = & \mathbf{f}(\mathbf{q}), & \mathbf{f}^-(\mathbf{q}) & = & \mathbf{0} & \text{if} & \lambda_m \ge 0 & \text{for all} \quad m = 1, \ldots, M \\ \mathbf{f}^+(\mathbf{q}) & = & \mathbf{0}, & \mathbf{f}^-(\mathbf{q}) & = & \mathbf{f}(\mathbf{q}) & \text{if} & \lambda_m \le 0 & \text{for all} \quad m = 1, \ldots, M \end{array}$$

Then use

$$\mathbf{F}(\mathbf{q}_L, \mathbf{q}_R) = \mathbf{f}^+(\mathbf{q}_L) + \mathbf{f}^-(\mathbf{q}_R)$$

Steger-Warming

Required $\mathbf{f}(\mathbf{q}) = \mathbf{A}(\mathbf{q})\,\mathbf{q}$

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Gives

$$\mathbf{f}(\mathbf{q}) = \mathbf{A}^+(\mathbf{q})\,\mathbf{q} + \mathbf{A}^-(\mathbf{q})\,\mathbf{q}$$

and the numerical flux

$$\mathbf{F}(\mathbf{q}_L, \mathbf{q}_R) = \mathbf{A}^+(\mathbf{q}_L)\,\mathbf{q}_L + \mathbf{A}^-(\mathbf{q}_R)\,\mathbf{q}_R$$

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Jacobians of the split fluxes are identical to $\mathbf{A}^\pm(\mathbf{q})$ only in linear case

$$\frac{\partial \mathbf{f}^\pm(\mathbf{q})}{\partial \mathbf{q}} = \frac{\partial\left(\mathbf{A}^\pm(\mathbf{q})\,\mathbf{q}\right)}{\partial \mathbf{q}} = \mathbf{A}^\pm(\mathbf{q}) + \frac{\partial \mathbf{A}^\pm(\mathbf{q})}{\partial \mathbf{q}}\,\mathbf{q}$$

Further methods: Van Leer FVS [Toro, 1999], AUSM [Wada and Liou, 1997]

MUSCL slope limiting

Monotone Upwind Schemes for Conservation Laws [van Leer, 1979]

$$\tilde{Q}^L_{j+\frac{1}{2}} = Q^n_j + \frac{1}{4}\left[(1-\omega)\,\Phi^+_{j-\frac{1}{2}}\Delta_{j-\frac{1}{2}} + (1+\omega)\,\Phi^-_{j+\frac{1}{2}}\Delta_{j+\frac{1}{2}}\right] ,$$

$$\tilde{Q}^R_{j-\frac{1}{2}} = Q^n_j - \frac{1}{4}\left[(1-\omega)\,\Phi^-_{j+\frac{1}{2}}\Delta_{j+\frac{1}{2}} + (1+\omega)\,\Phi^+_{j-\frac{1}{2}}\Delta_{j-\frac{1}{2}}\right]$$

with $\Delta_{j-1/2} = Q^n_j - Q^n_{j-1}$, $\Delta_{j+1/2} = Q^n_{j+1} - Q^n_j$.

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$$\Phi^+_{j-\frac{1}{2}} := \Phi\left(r^+_{j-\frac{1}{2}}\right) , \quad \Phi^-_{j+\frac{1}{2}} := \Phi\left(r^-_{j+\frac{1}{2}}\right) \quad \text{with} \quad r^+_{j-\frac{1}{2}} := \frac{\Delta_{j+\frac{1}{2}}}{\Delta_{j-\frac{1}{2}}} , \quad r^-_{j+\frac{1}{2}} := \frac{\Delta_{j-\frac{1}{2}}}{\Delta_{j+\frac{1}{2}}}$$

and *slope limiters*, e.g., *Minmod*

$$\Phi(r) = \max(0, \min(r, 1))$$

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and *slope limiters*, e.g., *Minmod*

$$\Phi(r) = \max(0, \min(r, 1))$$

Using a midpoint rule for temporal integration, e.g.,

$$Q^\star_j = Q^n_j - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(F(Q^n_{j+1}, Q^n_j) - F(Q^n_j, Q^n_{j-1})\right)$$

and constructing limited values from $Q^\star$ to be used in FV scheme gives a TVD method if

$$\frac{1}{2}\left[(1-\omega)\Phi(r) + (1+\omega)\, r\, \Phi\left(\frac{1}{r}\right)\right] < \min(2, 2r)$$

is satisfied for $r > 0$. Proof: [Hirsch, 1988]

Wave Propagation with flux limiting

Wave Propagation Method [LeVeque, 1997] is built on the flux differencing approach $\mathcal{A}^{\pm}\Delta := \hat{\mathbf{A}}^{\pm}(\mathbf{q}_L, \mathbf{q}_R)\Delta\mathbf{q}$ and the *waves* $\mathcal{W}_m := a_m \hat{\mathbf{r}}_m$, i.e.

$$\mathcal{A}^{-}\Delta\mathbf{q} = \sum_{\hat{\lambda}_m < 0} \hat{\lambda}_m \mathcal{W}_m \;, \quad \mathcal{A}^{+}\Delta\mathbf{q} = \sum_{\hat{\lambda}_m \geq 0} \hat{\lambda}_m \mathcal{W}_m$$

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Wave Propagation 1D:

$$\mathbf{Q}^{n+1} = \mathbf{Q}_j^n - \frac{\Delta t}{\Delta x}\left(\mathcal{A}^{-}\Delta_{j+\frac{1}{2}} + \mathcal{A}^{+}\Delta_{j-\frac{1}{2}}\right) - \frac{\Delta t}{\Delta x}\left(\tilde{\mathbf{F}}_{j+\frac{1}{2}} - \tilde{\mathbf{F}}_{j-\frac{1}{2}}\right)$$

with

$$\tilde{\mathbf{F}}_{j+\frac{1}{2}} = \frac{1}{2}\,|\mathcal{A}|\left(1 - \frac{\Delta t}{\Delta x}\,|\mathcal{A}|\right)\Delta_{j+\frac{1}{2}} = \frac{1}{2}\sum_{m=1}^{M}|\hat{\lambda}^m_{j+\frac{1}{2}}|\left(1 - \frac{\Delta t}{\Delta x}|\hat{\lambda}^m_{j+\frac{1}{2}}|\right)\tilde{\mathcal{W}}^m_{j+\frac{1}{2}}$$

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and wave limiter

$$\tilde{\mathcal{W}}^m_{j+\frac{1}{2}} = \Phi(\Theta^m_{j+\frac{1}{2}})\,\mathcal{W}^m_{j+\frac{1}{2}}$$

with

$$\Theta^m_{j+\frac{1}{2}} = \begin{cases} a^m_{j-\frac{1}{2}}/a^m_{j+\frac{1}{2}}\,, & \hat{\lambda}^m_{j+\frac{1}{2}} \geq 0\,, \\ a^m_{j+\frac{3}{2}}/a^m_{j+\frac{1}{2}}\,, & \hat{\lambda}^m_{j+\frac{1}{2}} < 0 \end{cases}$$

Wave Propagation Method in 2D

Writing $\tilde{\mathcal{A}}^{\pm}\Delta_{j\pm 1/2} := \mathcal{A}^{+}\Delta_{j\pm 1/2} + \tilde{\mathbf{F}}_{j\pm 1/2}$ one can develop a truly two-dimensional one-step method [Langseth and LeVeque, 2000]

$$
\begin{aligned}
\mathbf{Q}_{jk}^{n+1} = \mathbf{Q}_{jk}^{n} - \frac{\Delta t}{\Delta x_1}\Bigg(&\tilde{\mathcal{A}}^{-}\Delta_{j+\frac{1}{2},k} - \frac{1}{2}\frac{\Delta t}{\Delta x_2}\left[\mathcal{A}^{-}\tilde{\mathcal{B}}^{-}\Delta_{j+1,k+\frac{1}{2}} + \mathcal{A}^{-}\tilde{\mathcal{B}}^{+}\Delta_{j+1,k-\frac{1}{2}}\right] + \\
&\tilde{\mathcal{A}}^{+}\Delta_{j-\frac{1}{2},k} - \frac{1}{2}\frac{\Delta t}{\Delta x_2}\left[\mathcal{A}^{+}\tilde{\mathcal{B}}^{-}\Delta_{j-1,k+\frac{1}{2}} + \mathcal{A}^{+}\tilde{\mathcal{B}}^{+}\Delta_{j-1,k-\frac{1}{2}}\right]\Bigg) \\
- \frac{\Delta t}{\Delta x_2}\Bigg(&\tilde{\mathcal{B}}^{-}\Delta_{j,k+\frac{1}{2}} - \frac{1}{2}\frac{\Delta t}{\Delta x_1}\left[\mathcal{B}^{-}\tilde{\mathcal{A}}^{-}\Delta_{j+\frac{1}{2},k+1} + \mathcal{B}^{-}\tilde{\mathcal{A}}^{+}\Delta_{j-\frac{1}{2},k+1}\right] + \\
&\tilde{\mathcal{B}}^{+}\Delta_{j,k-\frac{1}{2}} - \frac{1}{2}\frac{\Delta t}{\Delta x_1}\left[\mathcal{B}^{+}\tilde{\mathcal{A}}^{-}\Delta_{j+\frac{1}{2},k-1} + \mathcal{B}^{+}\tilde{\mathcal{A}}^{+}\Delta_{j-\frac{1}{2},k-1}\right]\Bigg)
\end{aligned}
$$

that is stable for

$$
\left\{\max_{j\in\mathbb{Z}}|\hat{\lambda}_{m,j+\frac{1}{2}}|\frac{\Delta t}{\Delta x_1}, \max_{k\in\mathbb{Z}}|\hat{\lambda}_{m,k+\frac{1}{2}}|\frac{\Delta t}{\Delta x_2}\right\} \le 1\,, \quad \text{for all } m = 1,\ldots,M
$$

Further high-resolution methods

Some further high-resolution methods (good overview in [Laney, 1998]):

- FCT: 2nd order [Oran and Boris, 2001]

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3rd order methods must make use of strong-stability preserving Runge-Kutta methods [Gottlieb et al., 2001] for time integration that use a multi-step update

$$\tilde{\mathbf{Q}}_j^{\upsilon} = \alpha_\upsilon \mathbf{Q}_j^n + \beta_\upsilon \tilde{\mathbf{Q}}_j^{\upsilon-1} + \gamma_\upsilon \frac{\Delta t}{\Delta x} \left(\mathbf{F}_{j+\frac{1}{2}}(\tilde{\mathbf{Q}}^{\upsilon-1}) - \mathbf{F}_{j-\frac{1}{2}}(\tilde{\mathbf{Q}}^{\upsilon-1}) \right)$$

with $\tilde{\mathbf{Q}}^0 := \mathbf{Q}^n$, $\alpha_1 = 1$, $\beta_1 = 0$; and $\mathbf{Q}^{n+1} := \tilde{\mathbf{Q}}^{\Upsilon}$ after final stage Υ

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Typical storage-efficient SSPRK(3,3):

$$\tilde{\mathbf{Q}}^1 = \mathbf{Q}^n + \Delta t \mathcal{F}(\mathbf{Q}^n), \quad \tilde{\mathbf{Q}}^2 = \frac{3}{4}\mathbf{Q}^n + \frac{1}{4}\tilde{\mathbf{Q}}^1 + \frac{1}{4}\Delta t \mathcal{F}(\tilde{\mathbf{Q}}^1),$$
$$\mathbf{Q}^{n+1} = \frac{1}{3}\mathbf{Q}^n + \frac{2}{3}\tilde{\mathbf{Q}}^2 + \frac{2}{3}\Delta t \mathcal{F}(\tilde{\mathbf{Q}}^2)$$

Outline

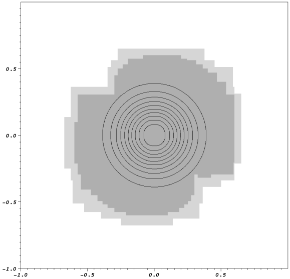
SAMR accuracy verification

Gaussian density shape

$$\rho(x_1, x_2) = 1 + e^{-\left(\frac{\sqrt{x_1^2+x_2^2}}{R}\right)^2}$$

is advected with constant velocities $u_1 = u_2 \equiv 1$, $p_0 \equiv 1$, $R = 1/4$

- Domain $[-1, 1] \times [-1, 1]$, periodic boundary conditions, $t_{end} = 2$
- Two levels of adaptation with $r_{1,2} = 2$, finest level corresponds to $N \times N$ uniform grid



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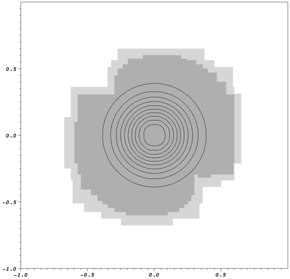


Use *locally* conservative interpolation

$$\check{\mathbf{Q}}^l_{v,w} := \mathbf{Q}^l_{ij} + f_1(\mathbf{Q}^l_{i+1,j} - \mathbf{Q}^l_{i-1,j}) + f_2(\mathbf{Q}^l_{i,j+1} - \mathbf{Q}^l_{i,j-1})$$

with factor $f_1 = \dfrac{x^v_{1,l+1} - x^i_{1,l}}{2\Delta x_{1,l}}$, $f_2 = \dfrac{x^w_{2,l+1} - x^j_{2,l}}{2\Delta x_{2,l}}$ to also test flux correction

This prolongation operator is not monotonicity preserving! Only applicable to smooth problems.

code/amroc/doc/html/apps/clawpack_2applications_2euler_22d_2GaussianPulseAdvection_2src_2Problem_8h_source.html

SAMR accuracy verification: results

VanLeer flux vector splitting with dimensional splitting, Minmod limiter

N	Unigrid		SAMR - fixup			SAMR - no fixup		
	Error	Order	Error	Order	$\Delta\rho$	Error	Order	$\Delta\rho$
20	0.10946400							
40	0.04239430	1.369						
80	0.01408160	1.590	0.01594820		0	0.01595980		2e-5
160	0.00492945	1.514	0.00526693	1.598	0	0.00530538	1.589	2e-5
320	0.00146132	1.754	0.00156516	1.751	0	0.00163837	1.695	-1e-5
640	0.00041809	1.805	0.00051513	1.603	0	0.00060021	1.449	-6e-5

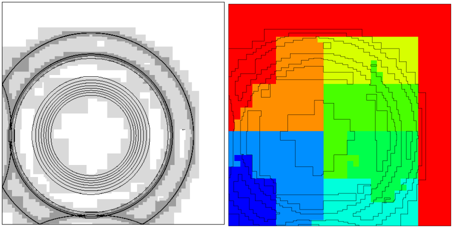
Fully two-dimensional Wave Propagation Method, Minmod limiter

N	Unigrid		SAMR - fixup			SAMR - no fixup		
	Error	Order	Error	Order	$\Delta\rho$	Error	Order	$\Delta\rho$
20	0.10620000							
40	0.04079600	1.380						
80	0.01348250	1.598	0.01536580		0	0.01538820		2e-5
160	0.00472301	1.513	0.00505406	1.604	0	0.00510499	1.592	5e-5
320	0.00139611	1.758	0.00147218	1.779	0	0.00152387	1.744	7e-5
640	0.00039904	1.807	0.00044500	1.726	0	0.00046587	1.710	6e-5

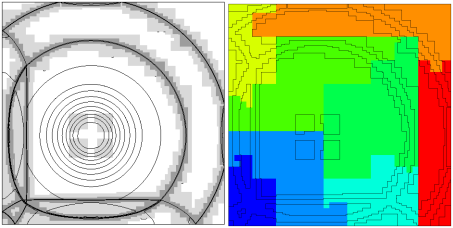
Benchmark run: blast wave in 2D

- 2D-Wave-Propagation Method with Roe's approximate solver
- Base grid 150×150
- 2 levels: factor 2, 4

Task [%]	$P\!=\!1$	$P\!=\!2$	$P\!=\!4$	$P\!=\!8$	$P\!=\!16$
Update by $\mathcal{H}^{(\cdot)}$	86.6	83.4	76.7	64.1	51.9
Flux correction	1.2	1.6	3.0	7.9	10.7
Boundary setting	3.5	5.7	10.1	15.6	18.3
Recomposition	5.5	6.1	7.4	9.9	14.0
Misc.	4.9	3.2	2.8	2.5	5.1
Time [min]	**151.9**	**79.2**	**43.4**	**23.3**	**13.9**
Efficiency [%]	**100.0**	**95.9**	**87.5**	**81.5**	**68.3**



After 38 time steps



After 79 time steps

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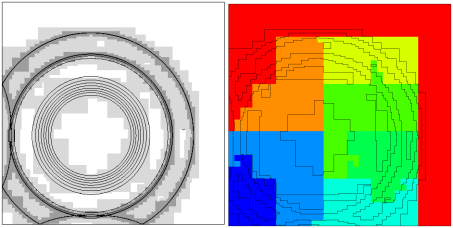

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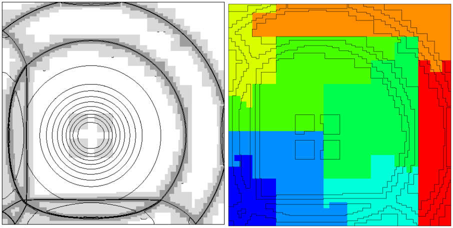

After 79 time steps

`code/amroc/doc/html/apps/clawpack_2applications_2euler_22d_2Box_2src_2Problem_8h_source.html`

Benchmark run 2: point-explosion in 3D

- Benchmark from the Chicago workshop on AMR methods, September 2003
- Sedov explosion - energy deposition in sphere of radius 4 finest cells
- 3D-Wave-Prop. Method with hybrid Roe-HLL scheme
- Base grid 32^3
- Refinement factor $r_l = 2$
- Effective resolutions: 128^3, 256^3, 512^3, 1024^3
- Grid generation efficiency $\eta_{tol} = 85\%$
- Proper nesting enforced
- Buffer of 1 cell

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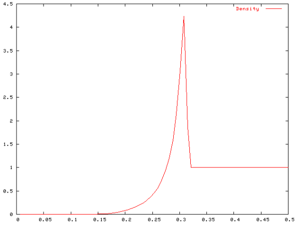
- Base grid 32^3
- Refinement factor $r_l = 2$
- Effective resolutions: 128^3, 256^3, 512^3, 1024^3
- Grid generation efficiency $\eta_{tol} = 85\%$
- Proper nesting enforced
- Buffer of 1 cell

$l_{\max} = 4$ solution



Benchmark run 2: point-explosion in 3D

- Benchmark from the Chicago workshop on AMR methods, September 2003
- Sedov explosion - energy deposition in sphere of radius 4 finest cells
- 3D-Wave-Prop. Method with hybrid Roe-HLL scheme
- Base grid 32^3
- Refinement factor $r_l = 2$
- Effective resolutions: 128^3, 256^3, 512^3, 1024^3
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- Proper nesting enforced
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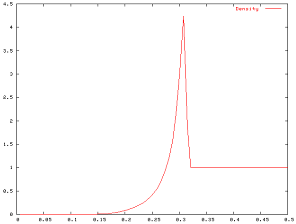
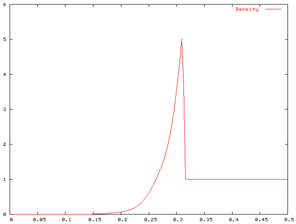
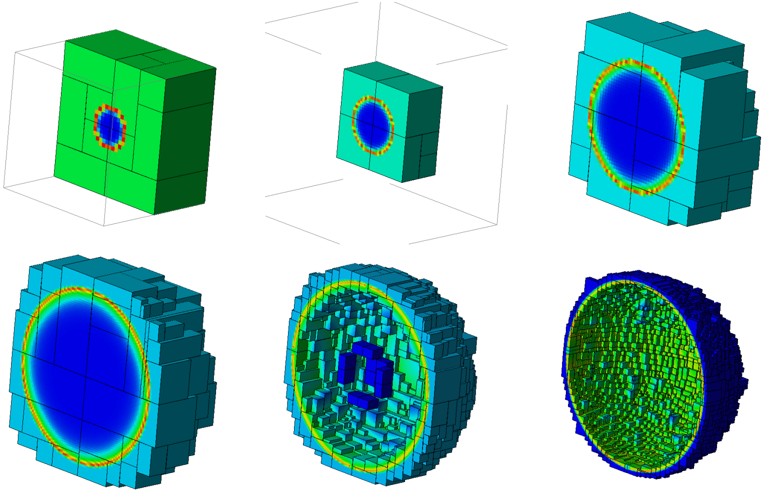


$l_{\max} = 5$ solution



Benchmark run 2: visualization of refinement

$l = 0$ $l = 1$ $l = 2$



$l = 3$ $l = 4$ $l = 5$

Benchmark run 2: performance results

Number of grids and cells

l	$l_{\max} = 2$		$l_{\max} = 3$		$l_{\max} = 4$		$l_{\max} = 5$	
	Grids	Cells	Grids	Cells	Grids	Cells	Grids	Cells
0	28	32,768	28	32,768	33	32,768	34	32,768
1	8	32,768	14	32,768	20	32,768	20	32,768
2	63	115,408	49	116,920	43	125,680	50	125,144
3			324	398,112	420	555,744	193	572,768
4					1405	1,487,312	1,498	1,795,048
5							5,266	5,871,128
$\sum$		180,944		580,568		2,234,272		8,429,624

Benchmark run 2: performance results

Number of grids and cells

l	$l_{max}=2$ Grids	$l_{max}=2$ Cells	$l_{max}=3$ Grids	$l_{max}=3$ Cells	$l_{max}=4$ Grids	$l_{max}=4$ Cells	$l_{max}=5$ Grids	$l_{max}=5$ Cells
0	28	32,768	28	32,768	33	32,768	34	32,768
1	8	32,768	14	32,768	20	32,768	20	32,768
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Breakdown of CPU time on 8 nodes SGI Altix 3000 (Linux-based shared memory system)

Task [%]	$l_{max}=2$		$l_{max}=3$		$l_{max}=4$		$l_{max}=5$	
Integration	73.7		77.2		72.9		37.8	
Fixup	2.6	*46*	3.1	*58*	2.6	*42*	2.2	*45*
Boundary	10.1	*79*	6.3	*78*	5.1	*56*	6.9	*78*
Recomposition	7.4		8.0		15.1		50.4	
Clustering	0.5		0.6		0.7		1.0	
Output/Misc	5.7		4.0		3.6		1.7	
Time [min]	0.5		5.1		73.0		2100.0	
Uniform [min]	5.4		160		∼5,000		∼180,000	
Factor of AMR savings	11		31		69		86	
Time steps	15		27		52		115	

Benchmark run 2: performance results

Number of grids and cells

l	$l_{max}=2$ Grids	$l_{max}=2$ Cells	$l_{max}=3$ Grids	$l_{max}=3$ Cells	$l_{max}=4$ Grids	$l_{max}=4$ Cells	$l_{max}=5$ Grids	$l_{max}=5$ Cells
0	28	32,768	28	32,768	33	32,768	34	32,768
1	8	32,768	14	32,768	20	32,768	20	32,768
2	63	115,408	49	116,920	43	125,680	50	125,144
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Factor of AMR savings	11		31		69		86	
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`code/amroc/doc/html/apps/clawpack_2applications_2euler_23d_2Sedov_2src_2Problem_8h_source.html`

Components

Directory `amroc/clawpack/src` contains generic Fortran functions:

- **?d/integrator_extended**: Contains an extended version of Clawpack 3.0 by R. LeVeque. The MUSCL approach was added, 3d fully implemented, interfaces have been adjusted for AMROC. These codes are equation independent.

  `code/amroc/doc/html/clp/files.html`

- **?d/equations**: Contains equation-specific Riemann solvers, flux functions as F77 routines.
- **?d/interpolation**: Contains patch-wise interpolation and restriction operators in F77.

Directory `amroc/clawpack` contains the generic C++ classes to interface the F77 library from ?d/integrator_extended with AMROC:

- **ClpIntegrator<VectorType, AuxVectorType, dim >**: Interfaces the F77 library from ?d/integrator_extended to Integrator<VectorType, dim>. Key function to fill is CalculateGrid().

  `code/amroc/doc/html/clp/classClpIntegrator_3_01VectorType_00_01AuxVectorType_00_012_01_4.html`

Components - II

- **ClpFixup<VectorType, FixupType, AuxVectorType, dim >**: The conservative flux correction is more complex in the waves of the flux difference splitting schemes. This specialization of AMRFixup<VectorType, FixupType, dim>considers this.

  `code/amroc/doc/html/clp/classClpFixup.html`

- A generic main program **amroc/clawpack/mains/amr_main.C** instantiates Integrator<VectorType, dim>, InitialCondition<VectorType, dim >, BoundaryConditions<VectorType, dim >

  `code/amroc/doc/html/clp/amr__main_8C.html`

- **Problem.h**: Allows simulation-specific alteration in class-libary style by derivation from predefined classes specified in **ClpStdProblem.h**

  `code/amroc/doc/html/clp/ClpStdProblem_8h.html`

- **ClpProblem.h**: General include before equation-specific C++ definition file is read that defines VectorType and provides Fortran function names required by amroc/amr/F77Interfaces classes

  `code/amroc/doc/html/clp/ClpProblem_8h_source.html code/amroc/doc/html/clp/euler2_8h.html`

Functions to link in Makefile.am

Interface objects from amroc/amr/F77Interfaces are used to mimic the interface of standard Clawpack, which constructs specific simulations by linking F77 functions. Required functions are:

- **init.f**: Initial conditions.
- **physbd.f**: Boundary conditions.
- **combl.f**: Initialize application specific common blocks.
- **$(EQUATION)/rp/rpn.f and rpt.f**: Equation-specific Riemann solvers in normal and transverse direction.
- **$(EQUATION)/rp/flx.f, $(EQUATION)/rp/rec.f**: Flux and reconstruction for MUSCL slope limiting (if used), otherwise dummy-routines/flx.f and dummy-routines/rec.f may be used.
- **$(EQUATION)/rp/chk.f**: Physical consistency check for debugging.
- **src.f**: Source term for a splitting method., otherwise dummy-routines/src.f can be linked.
- **setaux.f**: Set data in an additional patch-wise auxiliary array, otherwise dummy-routines/saux.f can be linked.

Outline

Favre-averaged Navier-Stokes equations

$$\frac{\partial \bar\rho}{\partial t} + \frac{\partial}{\partial x_n}\left(\bar\rho \tilde u_n\right) = 0$$

$$\frac{\partial}{\partial t}\left(\bar\rho \tilde u_k\right) + \frac{\partial}{\partial x_n}\left(\bar\rho \tilde u_k \tilde u_n + \delta_{kn}\bar p - \tilde\tau_{kn} + \sigma_{kn}\right) = 0$$

$$\frac{\partial \bar\rho \bar E}{\partial t} + \frac{\partial}{\partial x_n}\left(\tilde u_n(\bar\rho \bar E + \bar p) + \tilde q_n - \tilde\tau_{nj}\tilde u_j + \sigma_n^e\right) = 0$$

$$\frac{\partial}{\partial t}\left(\bar\rho \tilde Y_i\right) + \frac{\partial}{\partial x_n}\left(\bar\rho \tilde Y_i \tilde u_n + \tilde J_n^i + \sigma_n^i\right) = 0$$

with stress tensor

$$\tilde\tau_{kn} = \tilde\mu\left(\frac{\partial \tilde u_n}{\partial x_k} + \frac{\partial \tilde u_k}{\partial x_n}\right) - \frac{2}{3}\tilde\mu \frac{\partial \tilde u_j}{\partial x_j}\delta_{in} \; ,$$

heat conduction

$$\tilde q_n = -\tilde\lambda \frac{\partial \tilde T}{\partial x_n} \; ,$$

Favre-averaged Navier-Stokes equations

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$$\frac{\partial\bar{\rho}\bar{E}}{\partial t}+\frac{\partial}{\partial x_n}\left(\tilde{u}_n(\bar{\rho}\bar{E}+\bar{p})+\tilde{q}_n-\tilde{\tau}_{nj}\tilde{u}_j+\sigma_n^e\right)=0$$
$$\frac{\partial}{\partial t}\left(\bar{\rho}\tilde{Y}_i\right)+\frac{\partial}{\partial x_n}\left(\bar{\rho}\tilde{Y}_i\tilde{u}_n+\tilde{J}_n^i+\sigma_n^i\right)=0$$

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heat conduction

$$\tilde{q}_n=-\tilde{\lambda}\frac{\partial\tilde{T}}{\partial x_n}\ ,$$

and inter-species diffusion

$$\tilde{J}_n^i=-\bar{\rho}\tilde{D}_i\frac{\partial\tilde{Y}_i}{\partial x_n}$$

Favre-averaged Navier-Stokes equations

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$$\tilde{q}_n = -\tilde{\lambda}\frac{\partial \tilde{T}}{\partial x_n} \ ,$$

and inter-species diffusion

$$\tilde{J}_n^i = -\bar{\rho}\tilde{D}_i\frac{\partial \tilde{Y}_i}{\partial x_n}$$

Favre-filtering

$$\tilde{\phi} = \frac{\overline{\rho\phi}}{\bar{\rho}} \quad \text{with} \quad \bar{\phi}(\mathbf{x},t;\Delta_c) = \int_\Omega G(\mathbf{x}-\mathbf{x}';\Delta_c)\phi(\mathbf{x}',t)d\mathbf{x}'$$

Numerical solution approach

- Subgrid terms σ_{kn}, σ_n^e, σ_n^i are computed by Pullin's stretched-vortex model

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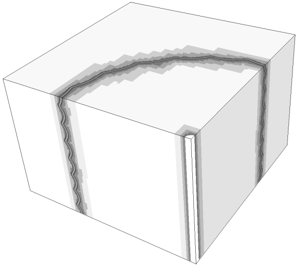
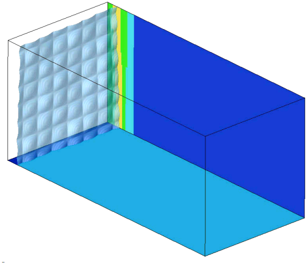
Example: Cylindrical Richtmyer-Meshkov instability

- Sinusoidal interface between two gases hit by shock wave
- Objective is correctly predict turbulent mixing
- Embedded boundary method used to regularize apex
- AMR base grid $95 \times 95 \times 64$ cells, $r_{1,2,3} = 2$
- $\sim 70,000\,\mathrm{h}$ CPU on 32 AMD 2.5GHZ-quad-core nodes



Planar Richtmyer-Meshkov instability

- Perturbed Air-SF6 interface shocked and re-shocked by Mach 1.5 shock
- Containment of turbulence in refined zones
- 96 CPUs IBM SP2-Power3
- WENO-TCD scheme with LES model
- AMR base grid $172 \times 56 \times 56$, $r_{1,2} = 2$, $10\,\mathrm{M}$ cells in average instead of $3\,\mathrm{M}$ (uniform)



Task	2ms (%)	5ms (%)	10ms (%)
Integration	45.3	65.9	52.0
Boundary setting	44.3	28.6	41.9
Flux correction	7.2	3.4	4.1
Interpolation	0.9	0.4	0.3
Reorganization	1.6	1.2	1.2
Misc.	0.6	0.5	0.5
Max. imbalance	1.25	1.23	1.30

`code/amroc/doc/html/apps/weno_2applications_2euler_23d_2RM_`
`_AirSF6_2src_2Problem_8h_source.html`

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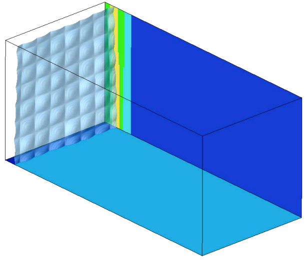

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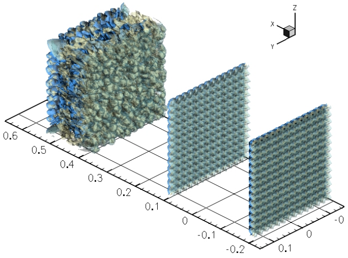

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Flux correction for Runge-Kutta method

Recall Runge-Kutta temporal update

$$\tilde{\mathbf{Q}}_j^{\upsilon} = \alpha_{\upsilon}\,\mathbf{Q}_j^n + \beta_{\upsilon}\,\tilde{\mathbf{Q}}_j^{\upsilon-1} + \gamma_{\upsilon}\,\frac{\Delta t}{\Delta x_k}\Delta\mathbf{F}^k(\tilde{\mathbf{Q}}^{\upsilon-1})$$

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rewrite scheme as

$$\mathbf{Q}^{n+1} = \mathbf{Q}^n - \sum_{\upsilon=1}^{\Upsilon} \varphi_\upsilon \frac{\Delta t}{\Delta x_k} \Delta \mathbf{F}^k(\tilde{\mathbf{Q}}^{\upsilon-1}) \quad \text{with} \quad \varphi_\upsilon = \gamma_\upsilon \prod_{\nu=\upsilon+1}^{\Upsilon} \beta_\nu$$

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Flux correction to be used [Pantano et al., 2007]

1. $\delta \mathbf{F}^{1,l+1}_{i-\frac{1}{2},j} := -\varphi_1 \mathbf{F}^{1,l}_{i-\frac{1}{2},j}(\tilde{\mathbf{Q}}^0)\,, \qquad \delta \mathbf{F}^{1,l+1}_{i-\frac{1}{2},j} := \delta \mathbf{F}^{1,l+1}_{i-\frac{1}{2},j} - \sum_{\upsilon=2}^{\Upsilon} \varphi_\upsilon \mathbf{F}^{1,l}_{i-\frac{1}{2},j}(\tilde{\mathbf{Q}}^{\upsilon-1})$
2. $\delta \mathbf{F}^{1,l+1}_{i-\frac{1}{2},j} := \delta \mathbf{F}^{1,l+1}_{i-\frac{1}{2},j} + \frac{1}{r_{l+1}^2} \sum_{m=0}^{r_{l+1}-1} \sum_{\upsilon=1}^{\Upsilon} \varphi_\upsilon \mathbf{F}^{1,l+1}_{v+\frac{1}{2},w+m} \left(\tilde{\mathbf{Q}}^{\upsilon-1}(t + \kappa \Delta t_{l+1})\right)$

Flux correction for Runge-Kutta method

Recall Runge-Kutta temporal update

$$\tilde{\mathbf{Q}}_j^{\upsilon} = \alpha_\upsilon \mathbf{Q}_j^n + \beta_\upsilon \tilde{\mathbf{Q}}_j^{\upsilon-1} + \gamma_\upsilon \frac{\Delta t}{\Delta x_k} \Delta \mathbf{F}^k(\tilde{\mathbf{Q}}^{\upsilon-1})$$

rewrite scheme as

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Storage-efficient SSPRK(3,3):

υ	α_υ	β_υ	γ_υ	φ_υ
1	1	0	1	$\frac{1}{6}$
2	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$
3	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$

Components

Directory `amroc/weno/src` contains the Fortran-90 solver library:

- **generic**: Implements the hybrid WENO-TCD method for Euler and Navier-Stokes equations, characteristic boundary conditions. Code uses F90 modules.

  `code/amroc/doc/html/weno/files.html`

- **equations**: Contains routines that specify between LES and laminar flow, the criterion for scheme hybridization, source terms handled by splitting.

Directory `amroc/weno` contains the generic C++ class to interface the F90 library with AMROC:

- **WENOIntegrator<VectorType, dim >**: Interfaces the F90 solver to Integrator<VectorType, dim>. CalculateGrid() is called separately for each stage of the Runge-Kutta time integrator.

  `code/amroc/doc/html/weno/classWENOIntegrator.html`

- **WENOFixup<VectorType, FixupType, dim >**: A specialized conservative flux correction that accumulates the correction terms throughout the stages of the Runge-Kutta time integrator.

  `code/amroc/doc/html/weno/classWENOFixup.html`

Components - II

- **WENOInterpolation<VectorType, InterpolationType, OutputType, dim >**: Is a quite elaborate data collection class based on AMRInterpolation<VectorType, dim>geared toward statistics processing typical for turbulent simulations. Has run-time function parser.

  `code/amroc/doc/html/weno/classWENOStatistics.html`

The interface otherwise follows the Clawpack integration closely:

- Generic main program **amroc/clawpack/mains/amr_main.C** is re-used.
- **Problem.h**: simulation-specific alteration of the C++ predefined classes specified in **WENOStdProblem.h**

  `code/amroc/doc/html/weno/WENOStdProblem_8h.html`

- **WENOProblem.h**: Include required C++ class definitions definitions, all Fortran function names defined in **WENOStdFunctions.h**

  `code/amroc/doc/html/weno/WENOProblem_8h_source.html` `code/amroc/doc/html/weno/WENOStdFunctions_8h.html`

- Interface objects from **amroc/amr/F77Interfaces** re-used and functions linked in Makefile.am as with Clawpack integrator

Outline

Governing equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot \left[\rho \mathbf{u}^{\mathbf{t}} \mathbf{u} + \left(p + \frac{\mathbf{B} \cdot \mathbf{B}}{2}\right) \mathbf{I} - \mathbf{B}^{\mathbf{t}} \mathbf{B}\right] = \mathbf{0}$$

$$\frac{\partial \rho E}{\partial t} + \nabla \cdot \left[\left(\rho E + p + \frac{\mathbf{B} \cdot \mathbf{B}}{2}\right) \mathbf{u} - (\mathbf{u} \cdot \mathbf{B})\, \mathbf{B}\right] = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot \left(\mathbf{u}^{\mathbf{t}} \mathbf{B} - \mathbf{B}^{\mathbf{t}} \mathbf{u}\right) = \mathbf{0}$$

with equation of state

$$p = (\gamma - 1)\left(\rho E - \rho \frac{\mathbf{u}^2}{2} - \frac{\mathbf{B}^2}{2}\right)$$

The ideal MDH model is still hyperbolic, yet by re-writing the induction equation, one finds that the magnetic field has to satisfy at all times t the elliptic constraint

$$\nabla \cdot \mathbf{B} = 0.$$

Generalized Lagrangian multipliers for divergence control

Hyperbolic-parabolic correction of 2d ideal MHD model [Dedner et al., 2002]:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_x}{\partial x} + \frac{\partial \rho u_y}{\partial y} = 0$$

$$\frac{\partial (\rho u_x)}{\partial t} + \frac{\partial}{\partial x}\left[\rho u_x^2 + p\left(p + \frac{\mathbf{B}\cdot\mathbf{B}}{2}\right) - B_x^2\right] + \frac{\partial}{\partial y}(\rho u_x u_y - B_x B_y) = 0$$

$$\frac{\partial (\rho u_y)}{\partial t} + \frac{\partial}{\partial x}(\rho u_x u_y - B_x B_y) + \frac{\partial}{\partial y}\left[\rho u_y^2 + p\left(p + \frac{\mathbf{B}\cdot\mathbf{B}}{2}\right) - B_y^2\right] = 0$$

$$\frac{\partial (\rho u_z)}{\partial t} + \frac{\partial}{\partial x}(\rho u_z u_x - B_z B_x) + \frac{\partial}{\partial y}(\rho u_z u_y - B_z B_y) = 0$$

$$\frac{\partial \rho E}{\partial t} + \frac{\partial}{\partial x}\left[\left(\rho E + p + \frac{\mathbf{B}\cdot\mathbf{B}}{2}\right)\mathbf{u_x} - (\mathbf{u}\cdot\mathbf{B})B_x\right] + \frac{\partial}{\partial y}\left[\left(\rho E + p + \frac{\mathbf{B}\cdot\mathbf{B}}{2}\right)\mathbf{u_y} - (\mathbf{u}\cdot\mathbf{B})B_y\right] = 0$$

$$\frac{\partial B_x}{\partial t} + \frac{\partial \psi}{\partial x} + \frac{\partial}{\partial y}(u_y B_x - B_y u_x) = 0$$

$$\frac{\partial B_y}{\partial t} + \frac{\partial}{\partial x}(u_x B_y - B_x u_y) + \frac{\partial \psi}{\partial y} = 0$$

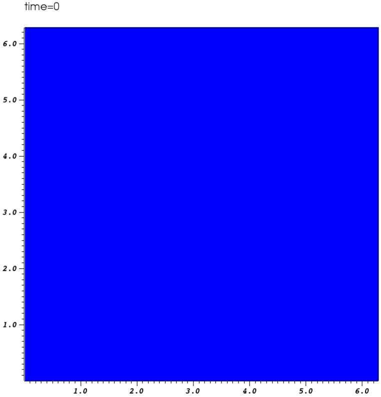
$$\frac{\partial B_z}{\partial t} + \frac{\partial}{\partial x}(u_x B_z - B_z u_x) + \frac{\partial}{\partial y}(u_y B_z - B_y u_z) = 0$$

$$\frac{\partial \psi}{\partial t} + c_h^2\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right) = -\frac{c_h^2}{c_p^2}\psi$$

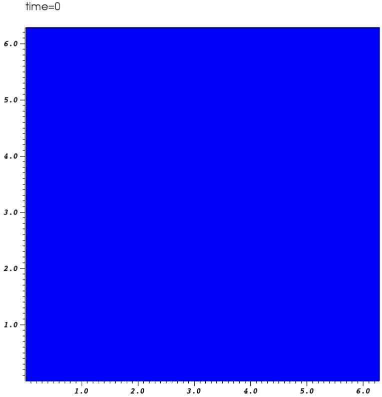
Orszag-Tang vortex

- Adaptive solution on 50×50 grid with 4 additional levels refined by $r_l = 2$
- Initial condition

$$\rho(x,y,0) = \gamma^2, \quad u_x(x,y,0) = -\sin(y), \quad u_y(x,y,0) = \sin(x), \quad u_z(x,y,0) = 0$$
$$p(x,y,0) = \gamma, \quad B_x(x,y,0) = -\sin(y), \quad B_y(x,y,0) = 2\sin(x), \quad B_z(x,y,0) = 0$$



Scaled gradient of ρ



Multi-resolution criterion with hierarchical thresholding

code/amroc/doc/html/apps/mhd_2applications_2eglm_22d_2OrszagTangVortex_2src_2Problem_8h_source.html

Orszag-Tang vortex

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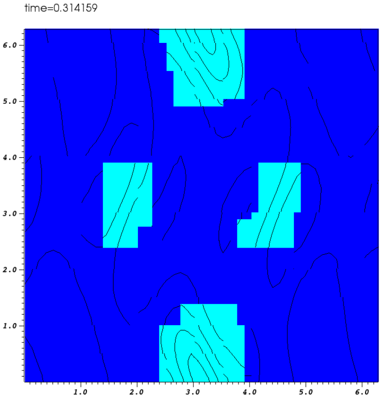
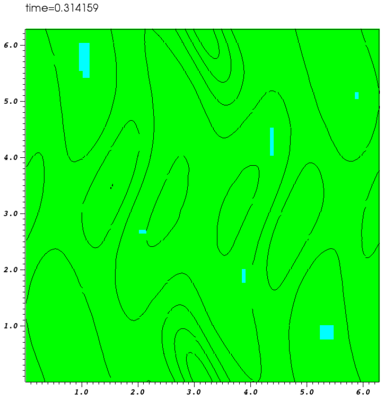


Scaled gradient of ρ



Multi-resolution criterion with hierarchical thresholding

code/amroc/doc/html/apps/mhd_2applications_2eglm_22d_2OrszagTangVortex_2src_2Problem_8h_source.html

Orszag-Tang vortex

- Adaptive solution on 50 × 50 grid with 4 additional levels refined by $r_l = 2$
- Initial condition

$$\rho(x,y,0) = \gamma^2, \quad u_x(x,y,0) = -\sin(y), \quad u_y(x,y,0) = \sin(x), \quad u_z(x,y,0) = 0$$

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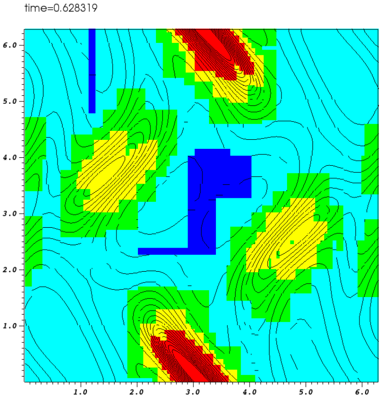
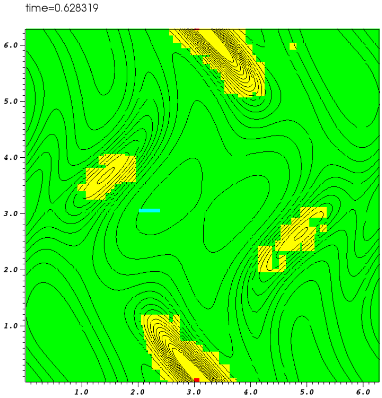


Scaled gradient of ρ



Multi-resolution criterion with hierarchical thresholding

code/amroc/doc/html/apps/mhd_2applications_2eglm_22d_2OrszagTangVortex_2src_2Problem_8h_source.html

Orszag-Tang vortex

- Adaptive solution on 50×50 grid with 4 additional levels refined by $r_l = 2$
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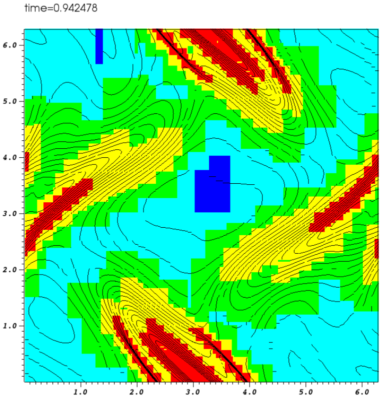
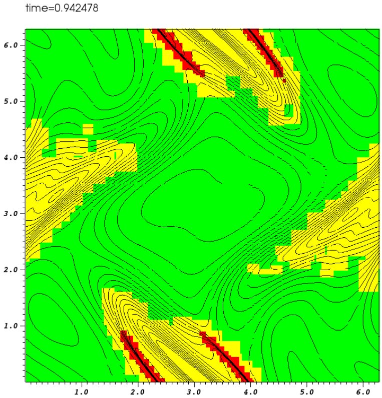


Scaled gradient of ρ



Multi-resolution criterion with hierarchical thresholding

code/amroc/doc/html/apps/mhd_2applications_2eglm_22d_2OrszagTangVortex_2src_2Problem_8h_source.html

Orszag-Tang vortex

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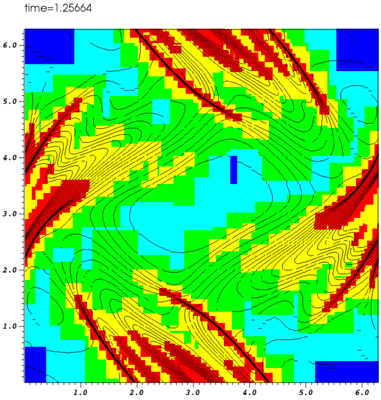

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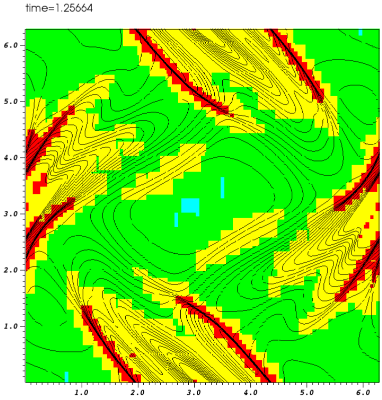

Multi-resolution criterion with hierarchical thresholding

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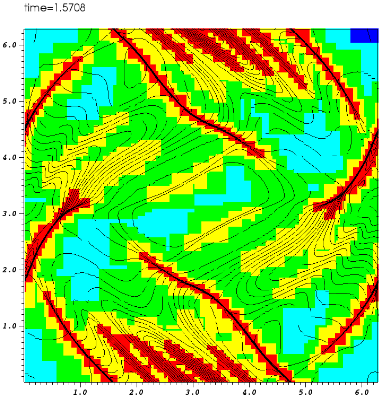

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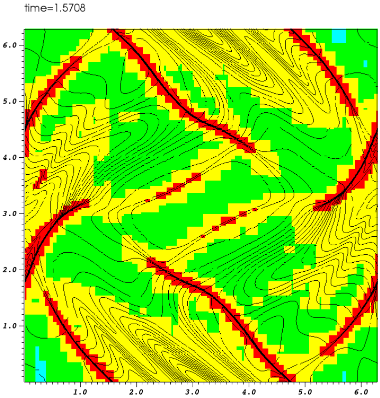

Multi-resolution criterion with hierarchical thresholding

`code/amroc/doc/html/apps/mhd_2applications_2eglm_22d_2OrszagTangVortex_2src_2Problem_8h_source.html`

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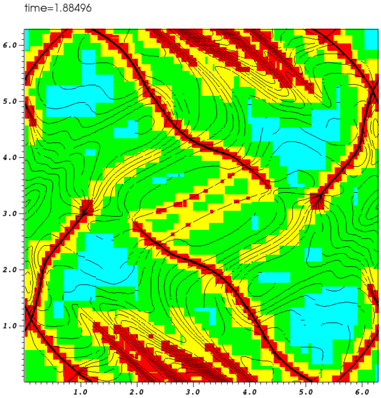
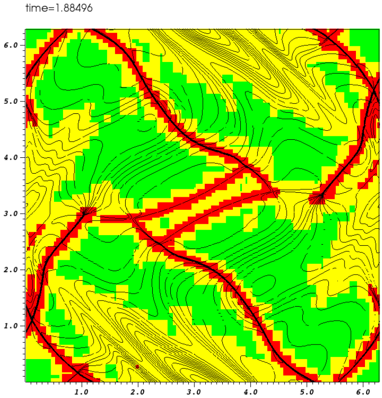


Scaled gradient of ρ



Multi-resolution criterion with hierarchical thresholding

code/amroc/doc/html/apps/mhd_2applications_2eglm_22d_2OrszagTangVortex_2src_2Problem_8h_source.html

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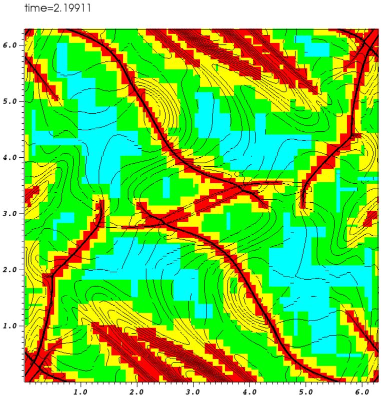
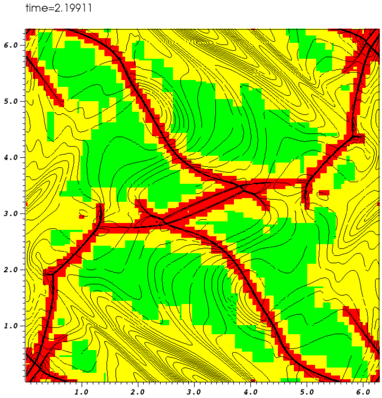


Scaled gradient of ρ



Multi-resolution criterion with hierarchical thresholding

code/amroc/doc/html/apps/mhd_2applications_2eglm_22d_2OrszagTangVortex_2src_2Problem_8h_source.html

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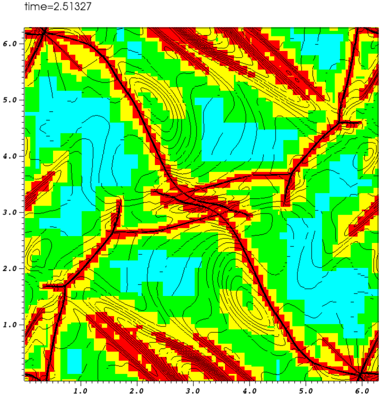
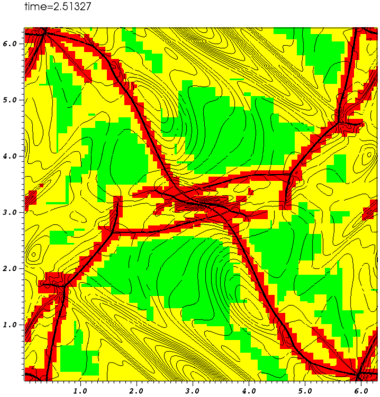


Scaled gradient of ρ



Multi-resolution criterion with hierarchical thresholding

code/amroc/doc/html/apps/mhd_2applications_2eglm_22d_2OrszagTangVortex_2src_2Problem_8h_source.html

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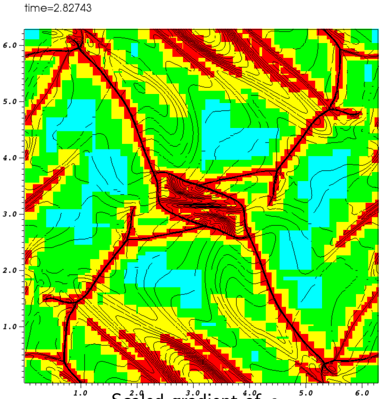
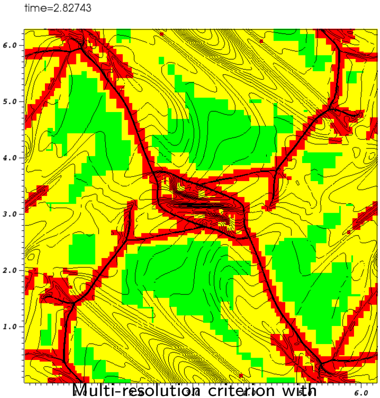


Scaled gradient of ρ



Multi-resolution criterion with hierarchical thresholding

code/amroc/doc/html/apps/mhd_2applications_2eglm_22d_2OrszagTangVortex_2src_2Problem_8h_source.html

Orszag-Tang vortex

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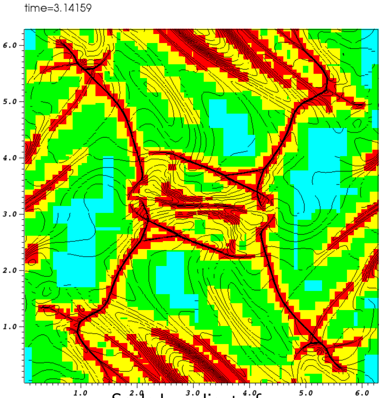
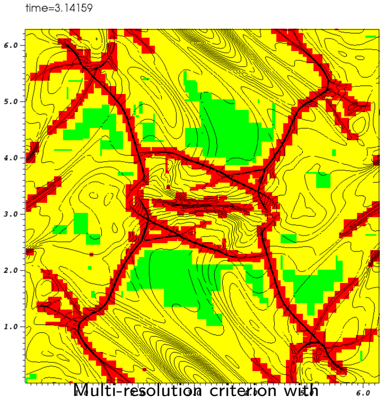


Scaled gradient of ρ



Multi-resolution criterion with hierarchical thresholding

`code/amroc/doc/html/apps/mhd_2applications_2eglm_22d_2OrszagTangVortex_2src_2Problem_8h_source.html`

Classes

Directory `amroc/mhd` contains the integrator that is in C++ throughout:

- **EGLM2D<DataType >**: GLM method in 2d plus standard initial and boundary conditions. Internal functions for flux evaluation, MUSCL resconstruction, etc. are member functions.

  `code/amroc/doc/html/mhd/classEGLM2D.html`

- Is derived from **SchemeBase<vector_type, dim >**, which is designed for the C++ interface classes in amroc/amr/Interfaces.

  `code/amroc/doc/html/amr/classSchemeBase.html`

- **amroc/amr/Interfaces**: Provides **SchemeIntegrator<SchemeType, dim >**, **SchemeInitialCondition<SchemeType, dim >**, **SchemeBoundaryCondition<SchemeType, dim >** and further interfaces that use classes derived from **SchemeBase<vector_type, dim >** as template parameter. This provides a single-class location for new schemes in C++.

  `code/amroc/doc/html/amr/classSchemeIntegrator.html` `code/amroc/doc/html/amr/classSchemeInitialCondition.html`

- **Problem.h**: Specific simulation is defined in Problem.h only. Predefined classes specified in **MHDStdProblem.h** and **MHDProblem.h** similar but simpler as before.

  `code/amroc/doc/html/mhd/MHDStdProblem_8h.html` `code/amroc/doc/html/mhd/MHDProblem_8h_source.html`

Further hyperbolic solvers

- **amroc/rim**: Riemann invariant manifold method. 2d implementation in F77 with straightforward integration into AMROC.

  `code/amroc/doc/html/rim/files.html`

- **amroc/balans**: 2nd order accurate central difference scheme. 2d implementation in F77 and excellent template for Fortran scheme incorporation into AMROC.

  `code/amroc/doc/html/balans/files.html`

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