A LAGRANGEAN/SURROGATE APPROACH TO p-MEDIAN PROBLEMS

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Abstract

The p-median problem is the problem of locating p facilities (medians) on a network so as to minimize the sum of all the distances from each demand point to its nearest facility. A successful approach to approximately solve this problem is the use of Lagrangean heuristics, based upon Lagrangean relaxation and subgradient optimization. We propose in this paper a Lagrangean/surrogate heuristic approach to p-median problems. Lagrangean and surrogate relaxations are combined relaxing in the surrogate way the assignment constraints in the p-median formulation. Then, the Lagrangean relaxation of the surrogate constraint is obtained and approximately optimized (one-dimensional dual). Lagrangean/surrogate relaxations are very stable (low-oscillating) and reach the same good results of Lagrangean (alone) heuristics in less computational times. The paper presents several computational tests which have been conducted with problems from the literature, a set of instances presenting large duality gaps and a set of time consuming instances.

Resumo

O problema das p-medianas pode ser descrito como o de localizar p facilidades (medianas) em uma rede minimizando a soma de todas as distâncias de cada ponto de demanda a sua mediana mais próxima. A relaxaçãop Lagrangeana tem sido usada com sucesso, combinada com heuristicas, para obter soluções aproximadas do problema. Propõe-se neste trabalho a abordagem de heuristica Lagrangeana/surrogate ao problema das p-medianas. As relaxações Lagrangeana e surrogate são combinadas relaxando inicialmente de forma surrogate as restrições de atribuição na formulação do problema. Em seguida, a relaxação Lagrangeana da restrição surrogate é obtida e aproximadamente otimizada (dual uni-dimensional). A relaxação Lagrangeana/surrogate é estavel e produz os mesmos bons limites da Lagrangeana em tempo computacional reduzido. Este trabalho apresenta testes computacionais com ambas as relaxações, usando instâncias de problemas testes diponíveis na literatura, um conjunto de instâncias que apresentam grande "gap" de dualidade e outro formado por intâncias que consomem grande tempo compuatcional.

Keywords: *Lagrangean and surrogate relaxation, Location problems, P-median problems.*

1. Introduction

The search for p-median nodes on a network is a classical location problem. The objective is to locate p facilities (medians) so as to minimize the sum of the distances from each demand point to its nearest facility.

Hakimi [11,12] was the first to formulate the problem for locating a single and multi-medians. He also proposed a simple enumeration procedure to solve the problem. The problem is well-known to be NP-hard [9]. Several heuristics have been developed for p-median problems. Some of them are used to obtain good initial solutions or to calculate intermediate solutions on search tree nodes. Teitz and Bart [23] proposed simple interchange heuristics (see also Maranzana [19]). More complete approaches explore a search tree. They appeared in Efroymson and Ray [5], Jarniven and Rajala [15], Neebe [21], Christofides and Beasley [3], Beasley [2] and Galvão and Raggi [8]. The combined used of Lagrangean relaxation and subgradient optimization in a primal-dual viewpoint was found to be a good solution approach to the problem [2,3,8].

Beasley [2] describes very effective heuristics for a class of location problems. They are called Lagrangean heuristics, and use Lagrangean relaxation and subgradient optimization. At each subgradient iteration, Lagrangean solutions are made primal feasible by applying simple heuristics, that are followed by interchange heuristics after a number of iterations. Lorena and Narciso [18] introduced relaxation heuristics for generalized assignment problems, using a generalized subgradient algorithm. The new relaxation presented is a surrogate relaxation that was used before in other applications, such as set covering problems [15] and multidimensional knapsack problems [6]. For the problems and instances studied, the performance of surrogate heuristics to find near-optimal solutions reduced computations times by a factor of two, relative to the corresponding Lagrangean heuristics with similar bounds.

The objective of this work is to compare Lagrangean/surrogate and Lagrangean (alone) for p-median problems. The Lagrangean/surrogate combines the two well-known Lagrangean and surrogate relaxation for the p-median problem. The relaxations are combined relaxing in the surrogate way the assignment constraints in the p-median formulation. Then, the Lagrangean relaxation of the surrogate constraint is obtained and approximately optimized (one-dimensional dual). Lagrangean/surrogate relaxations are very stable (low-oscillating) and reach the same good results of Lagrangean (alone) heuristics in less computational times. The set of test problems is divided in two, one with small problems presenting large dual gaps and other with the (hard) time consuming instances of the OR-library [1], i.e., the instances for which the number of medians is about 1/3 of the number of nodes.

In section two we present the relaxation used and some theory to explain their good behavior. Section three details the general subgradient heuristic. The computational tests which have been conducted with problems from the literature are presented in the next section. We conclude confirming that the Lagrangean/surrogate heuristic is better than the Lagrangean alone heuristic for time consuming p-median instances.

2. The Lagrangean/Surrogate Relaxation

The p-median problem considered in this paper is modeled as the following binary integer programming problem:

$$v(P) \;=\; \min \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}x_{ij}$$

(P) subject to

$$\sum_{i=1}^{n} x_{ij} = 1;\; j \in N \qquad (1)$$

$$\sum_{i=1}^{n} x_{ii} = p \qquad (2)$$

$$x_{ij} \le x_{ii};\; i, j \in N \qquad (3)$$

$$x_{ij} \in \{0,1\};\; i, j \in N \qquad (4)$$

where:

$[d_{ij}]_{nxn}$ is a symmetric cost (distance) matrix, with $d_{ii} = 0$, $\forall$ i;
$[x_{ij}]_{nxn}$ is the allocation matrix, with $x_{ij} = 1$ if node i is allocated to node j, and $x_{ij} = 0$, otherwise; $x_{ii} = 1$ if node i is a median and $x_{ii} = 0$, otherwise;
p is the number of facilities (medians) to be located;
n is the number of nodes in the network, and N = {1, ..., n}.

Constraints (1) and (3) ensure that each node j is allocated to only one node i , which must be a median. Constraint (2) determines the exact number of medians to be located (p), and (4) gives the integer conditions.

We use here relaxation heuristics to approximately solve problem (P). The surrogate and Lagrangean/surrogate relaxation are presented as follows.

As proposed by Glover [10], for a given $\lambda \in R_+^m$, a surrogate relaxation of (P) can be defined by:

$$v(SP^{\lambda}) \;=\; \min \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}x_{ij}$$

(SP^{λ}) subject to

$$\sum_{j=1}^{n}\sum_{i=1}^{n} \lambda_j x_{ij} = \sum_{j=1}^{n} \lambda_j \qquad (5)$$

and (2), (3) and (4).

The optimal value $v(SP^{\lambda})$ is less than or equal to v(P), and its best value can result in a surrogate dual $\max_{\lambda\ge 0} v(SP^{\lambda})$. The surrogate function s: $R_+^m \to R$, (λ, $v(SP^{\lambda})$) has some properties that make it difficult to find a dual solution. Some methods proposed in the literature find the approximate solution of the surrogate dual, such that of Dyer [4] and Karwan and Rardin [16]. Note here that problem (SP^{λ}) can not be easily solved, as it is an integer linear problem with no special structure to explore. See [22] for a book describing Lagrangean and surrogate relaxations.

Due to the difficulties with relaxation (SP^{λ}) we proposed to relax again the problem, now in the Lagrangean way. For a given $t \ge 0$, constraint (5) is relaxed, and the Lagrangean/surrogate relaxation is given by:

$$v(L_tSP^{\lambda}) = \min \sum_{j=1}^{n} \sum_{i=1}^{n} (d_{ij} - t\lambda_j)x_{ij} + t\sum_{j=1}^{n} \lambda_j$$

(L_tSP^{λ})

subject to (2), (3) and (4).

For given $t \geq 0$ and $\lambda \in R_+^m$, $v(L_tSP^{\lambda}) \leq v(SP^{\lambda}) \leq v(P)$. (L_tSP^{λ}) is solved considering implicitly constraint (2) and decomposing for index i , obtaining the following n problems

$$\min \sum_{j=1}^{n} (d_{ij} - t\lambda_j)x_{ij}$$

subject to (3) and (4).

Each problem is easily solved letting

$$\beta_i = \sum_{j=1}^{n} \{ \min (0 , d_{ij} - t\lambda_j) \}, \qquad (6)$$

and choosing I as the index set of the p smallest β_i (here constraint (2) is considered implicitly). Then, a solution x_{ij}^{λ} to problem (L_tSP^{λ}) is:

$$x_{ii}^{\lambda} = \begin{cases} 1, & if \ i \in I \\ 0, & otherwise \end{cases}$$

and for all $i \neq j$

$$x_{ij}^{\lambda} = \begin{cases} 1, & if \ i \in I \quad and \quad d_{ij} - t\lambda_j < 0 \\ 0, & otherwise \end{cases}$$

The Lagrangean/surrogate solution is given by:

$$v(L_tSP^{\lambda}) = \sum_{i=1}^{n} \beta_i \, x_{ii} + t\sum_{j=1}^{n} \lambda_j$$

The interesting characteristic of relaxation (L_tSP^{λ}), is that for $t = 1$ we have the usual Lagrangean relaxation using the multiplier λ. For a fixed multiplier λ , the best value for t can be found by solving a Lagrangean dual:

$$(D_t^{\lambda}) \qquad v(D_t^{\lambda}) = \max_{t \geq 0} v(L_tSP^{\lambda}).$$

It is immediate that $v(SP^{\lambda}) \geq v(D_t^{\lambda}) \geq v(L_1SP^{\lambda})$. It is well-known that the Lagrangean function

$l:R_+ \rightarrow R$, $(t,v(L_tSP^{\lambda}))$, is concave and piecewise linear [22]. The best Lagrangean/surrogate relaxation value gives an improved bound to the usual Lagrangean relaxation. An exact solution to (D_t^{λ}) may be obtained by a search over different values of t (see Minoux [20] and Handler and Zang [13]). However, in general, we have an interval of values $t_0 \leq t \leq t_1$ (with $t_0 = 1$ or $t_1 = 1$) which also produces improved bounds (see Figure 1, for the case $t_1 = 1$).

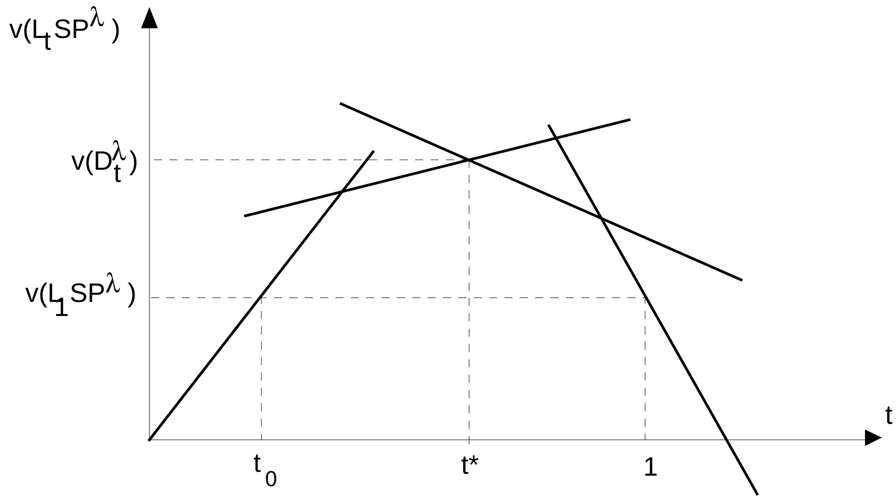

Figure 1: Lagrangean/surrogate bounds.

So, in order to obtain an improved bound to the usual Lagrangean relaxation it is not necessary to find the best value t^* , as is enough to find a value T such as $t_0 \le T \le t_1$. To find the approximate best Lagrangean/surrogate multiplier T we have used the following search procedure:

Search Heuristic (SH)

Given S (the initial step size) and M (the maximum number of iterations) let t be the current Lagrangean/surrogate multiplier, s be the current step size, k be the current number of iterations, and z be the current best lower bound.

Set *$t = 0$, $s = S$, $z = 0$, $k = 0$;*
Repeat
   Update *t and n using $t = t + s$ and $k = k + 1$;*
   If *$k > M$* ***then stop***
   Else *solve $(L_t SP^\lambda)$*
   End_If
   If *$v(L_t SP^\lambda) > z$* ***then*** *do the following:*
      Set *$T = t$ and $z = v(L_t SP^\lambda)$;*
      Calculate *the slope ω^λ of the Lagrangean/surrogate dual function using*

$$\omega^\lambda = \sum_{j=1}^{n} \lambda_j - \sum_{j=1}^{n} \sum_{i \in I} \lambda_j x_{ij} ;$$

      If *$\omega^\lambda < 0$* ***then*** *try to improve the current multiplier solving $(L_{t-s/2} SP^\lambda)$, updating T if necessary and* ***stop***
      End_If
   Else
      Try to improve the current multiplier solving $(L_{t-s/2} SP^\lambda)$, updating T if necessary;
      If *$n \le \lceil M/2 \rceil$* ***then*** *the initial step size is too large and must be halved for the next procedure applications so update S by setting $S = S/2$;*
      Stop
      End_If

End_if
Until *(stop conditions).*

3. The General Subgradient Heuristic

The following general subgradient algorithm is used as a base to the relaxation heuristics proposed in this work. In this algorithm the sets O and C are defined as: O = { i | x_{ii} = 1 } and C = { i | x_{ii} = 0 }, that is, O is the set of nodes already fixed as medians and C is the set of nodes already fixed as non-medians.

General Subgradient Heuristic (GSH)

Given $\lambda \geq 0, \lambda \neq 0$;
Set $lb = -\infty$, $ub = +\infty$, $O = \emptyset$, $C = \emptyset$;
Repeat
Solve *relaxation* (R^{λ}) *(Lagrangean or Lagrangean/surrogate)*
obtaining x^{λ} *and* $v(R^{\lambda})$;
Obtain *a feasible solution* x_f *and* $v_f = \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} x_{f\,ij}$;
Update $lb = \max\,[lb, v(R^{\lambda})]$;
Update $ub = \min\,[ub, v_f]$;
Fix x_{ii} *at the value* α $(\alpha \in \{0,1\})$ *if* $v(R^{\lambda} / x_{ii} = 1 - \alpha) \geq ub$, $i \in N - (O \cup C)$;
Update *the sets O and C accordingly;*
Set $g_j^{\lambda} = 1 - \sum_{i=1}^{n} x_{ij}^{\lambda}$, $j \in N$;
Update *the step size* θ;
Set $\lambda_j = \max\,\{ 0, \lambda_j + \theta . g_j^{\lambda} \}$, $j \in N$;
Until *(stopping tests).*

GSH have some early successful applications to generalized assignment problems [18], set covering problems [17] and 0-1 multiknapsack problems [6].

Two versions of GSH have been tested and the computational results are reported in the next section. They differ on the relaxation (R^{λ}) used. The first algorithm uses the well-know Lagrangean relaxation $(R^{\lambda}) = (L_1SP^{\lambda})$, and reproduces the Lagrangean heuristic of [2].

The second algorithm uses the new Lagrangean/surrogate relaxation $(R^{\lambda}) = (L_TSP^{\lambda})$, where T is the approximately best value for t obtained by the procedure SH described in section two. SH results in a multiplier T which is used in the Lagrangean/surrogate relaxation. However, if the search procedure produces the same multiplier T for P consecutive iterations of GSH, then the next Lagrangean/surrogate relaxations will use this fixed value T as the multiplier and the search is no demand nodeperformed. In this work we have used the following parameter values in SH: S = 0.5, M = 5 and P = 10.

For both algorithms the initial λ used is $\lambda_j = \min_{i \in N} \{ d_{ij} \}$, $j \in N$. The step sizes used are:

$\theta = \pi \, (ub - lb) / \| g^{\lambda} \|^2$. The control of parameter π is the Held and Karp [14] classical control. It makes $0 \le \pi \le 2$, beginning with $\pi = 2$ and halving π whenever lb does not increase for 30 successive iterations.

The stopping tests used are:

a) number of iterations greater than 1000;
b) $\pi \le 0.005$;
c) ub - lb < 1.

Solution x^{λ} is not necessarily feasible to (P), but the set I identifies median nodes that can be used to produce feasible solutions to (P). Two heuristics are used to make x^{λ} primal feasible. The first calculates the upper bound at each iteration of GSH while π is not halved. This heuristic simply makes:

$$v_f = \sum_{j=1}^{n} \left(\min_{i \in I} d_{ij} \right)$$

The second, as suggested by Beasley [2], is an interchange heuristic which is used when π is updated to $\pi/2$. Considering that in expression (6) the β_i ($i \in N$) are sorted in ascending order, this heuristic applies the following procedure:

Interchange Heuristic (IH)

Set $U = \sum_{j=1}^{n} \left(\min_{i \in I} d_{ij} \right)$ *corresponding to the solution* x^{λ} *associated with the current maximum lower bound lb.*

Set *m = max (25, n/10);*
m = min (m, n-p);
For $j = p+1$ *to* $p+m$; $j \notin C$ ***do***
For $i = 1$ *to* p ; $i \notin O$ ***do***
Interchange β_i *with* β_j*, updating I accordingly;*
$$v_f = \sum_{j=1}^{n} \left(\min_{i \in I} d_{ij} \right)$$
If $v_f < U$ ***then*** $U = v_f$
Else *interchange* β_i *with* β_j *and update I*
End_If
End_For
End_For
If $U < ub$ ***then*** $ub = U$
End_If

4. Computational Tests

The Lagrangean and Lagrangean/surrogate heuristics discussed above were programmed in C and run on a IBM Risc/6000 model 3AT workstation (compiled using xlc compiler with -O2 optimization option).

An initial set of instances used for the tests are obtained from the work of Galvao et al. [7], and although small (n = 100 and n = 150), the instances present duality gaps larger than to 1% for some values of p (number of medians). They can be considered hard instances for Lagrangean approaches in the sense of duality gaps.

The other set of instances are drawn from OR-Library [1], and can be considered easy problems for Lagrangean approaches in the sense of duality gaps. The gaps can be all closed [2].

For this work the objective is to show that Lagrangean/surrogate are better than Lagrangean (alone) heuristics in computational times performance. The first set of instances [7] are included only to show that the duality gaps performance of both relaxations are comparable, even in presence of large gaps. The second set of instances was used for computational times comparison. The time consuming instances of the OR-Library set are those presenting ratios n/p approximately equal to 3 , and are then selected for the tests. The instances (n = 700, p = 233), (n = 800, p = 267) and (n = 900, p = 300) were not considered in the OR-Library, and their optimal values were obtained running the Lagrangean/surrogate heuristic without limit of iterations searching for the optimality condition
ub - lb < 1.

The results are reported in the tables below (all computer times shown exclude the time needed to setup the problem).

Tables 1 and 2 show the results obtained by Lagrangean heuristic and Lagrangean/surrogate heuristic, respectively for the Galvao et al. [7] and OR-Library instances. Each table contains:

a) gap1 := (100 * [ub - optimal] / optimal), is the percentage deviation from optimal to the best feasible solution value found by the corresponding heuristic procedure;

b) gap2 := (100 * [optimal - lb] / optimal), is the percentage deviation from optimal to the best relaxation value found by the corresponding heuristic procedure; and

c) The ratio between the total computational times, Lagrangean/surrogate by Lagrangean (in IBM Risc/6000 seconds).

From Tables 1 and 2 we can see that in terms of duality gaps, Lagrangean/surrogate heuristics reach the same (good) results of Lagrangean (alone) heuristics. The total computational times are almost the same for the small instances of table 1, but in table 2 the Lagrangean/surrogate heuristic was able to find the same results using in mean 80% of time needed by the Lagrangean (alone) heuristic.

				Lagrangean		Lagrangean/Surrogate		Ratio
Prob.	***n***	***p***	***Optimal solution***	***gap1***	***gap2***	***gap1***	***gap2***	***Time(LS)/ Time(L)***
1	100	5	5703	-	0,340	-	0,346	1,12
2		10	4426	1,469	3,746	2,327	3,728	1,11
3		15	3893	0,899	0,900	0,308	0,895	1,01
4		20	3565	0,084	0,089	-	0,093	1,00
5		25	3291	-	0,061	-	0,067	1,05
6		30	3032	0,066	0,065	-	0,056	1,09
7		40	2542	-	-	-	-	1,08
8		50	2083	-	-	-	-	0,63
9	150	5	10839	-	1,402	-	1,404	0,94
10		10	8729	0,378	3,154	0,722	3,158	0,87
11		15	7390	3,315	4,917	1,922	4,906	1,02
12		20	6454	3,099	2,978	3,037	2,975	1,11
13		25	5875	1,617	1,015	1,191	1,009	1,12
14		30	5495	1,292	0,212	0,928	0,208	1,01
15		40	4907	0,102	0,065	0,061	0,068	1,02
16		50	4374	-	0,068	-	0,062	1,02
	Average Values			0,770	1,188	0,656	1,186	1,01

Table 1: Computational results for the Galvao et al. [7] instances.

				Lagrangean		Lagrangean/Surrogate		Ratio
Prob.	***n***	***p***	***Optimal solution***	***gap1***	***gap2***	***gap1***	***gap2***	***Time(LS)/ Time(L)***
1	100	33	1355	-	-	-	-	0,502
2	200	67	1255	-	-	-	-	0,723
3	300	100	1729	-	-	-	-	0,872
4	400	133	1789	-	-	-	-	1,268
5	500	167	1828	-	-	-	-	0,859
6	600	200	1989	-	-	-	-	0,782
7	700	233	1847	-	-	-	-	0,634
8	800	267	2035	-	-	-	-	0,629
9	900	300	2106	0,047	0,003	0,807	0,001	0,996
	Average Values			0,003	0,000	0,050	0,000	0,807

Table 2: Computational results for the time consuming instances of OR-Library [1]

The total time is largely influenced by the stop tests, and cannot serve as a controlled measure. We decided to control the running times for the instances in table 2 collecting the times necessary to gap2 be at least some pre-fixed values (9, 7, 5, 3 and 1). Table 3 shows the time ratios for gap2 to reach the fixed values. The ratios consider Lagrangean/surrogate by Lagrangean times. The average values show that the Lagrangean/surrogate used only 38 to 54% of the times needed by the corresponding Lagrangean. For time consuming instances it can be very interesting. For example, the running time used by the Lagrangean heuristic to reach gap2 = 1 using the instance (600,200) (one with the worst ratio in table 3) was approximately 1120 seconds, while the Lagrangean/surrogate used only 672 seconds.

		Time ratios (LS/L) for gap 2 = fixed value					
n	***p***	***gap2 = 9***	***gap2 = 7***	***gap2 = 5***	***gap2 = 3***	***gap2 = 1***	***Total***
100	33	0,66	0,45	0,45	0,45	0,55	0,502
200	67	0,40	0,41	0,41	0,71	0,57	0,723
300	100	0,44	0,44	0,44	0,35	0,57	0,872
400	133	0,39	0,39	0,39	0,54	0,43	1,268
500	167	0,38	0,38	0,38	0,53	0,62	0,859
600	200	0,36	0,36	0,36	0,51	0,61	0,782
700	233	0,34	0,34	0,34	0,26	0,51	0,634
800	267	0,34	0,34	0,34	0,35	0,51	0,629
900	300	0,34	0,34	0,34	0,34	0,50	0,996
	Average Values	0,41	0,38	0,38	0,45	0,54	0,807

Table 3: CPU time the time ratios for gap2 reach some fixed values.

In order to compare the computational behavior of Lagrangean and Lagrangean/surrogate heuristics we have plotted (see Figure 2) the values of $v(L_tSP^{\lambda})$ obtained at each iteration from these heuristics for problem n = 600 and p = 200. We can observe that the sequence of Lagrangean/surrogate relaxations are more stable than the corresponding Lagrangean ones. The local searches in SH at the first iterations of GSH accelerated the overall converge of the Lagrangean/surrogate, although without loss of quality in duality bounds.

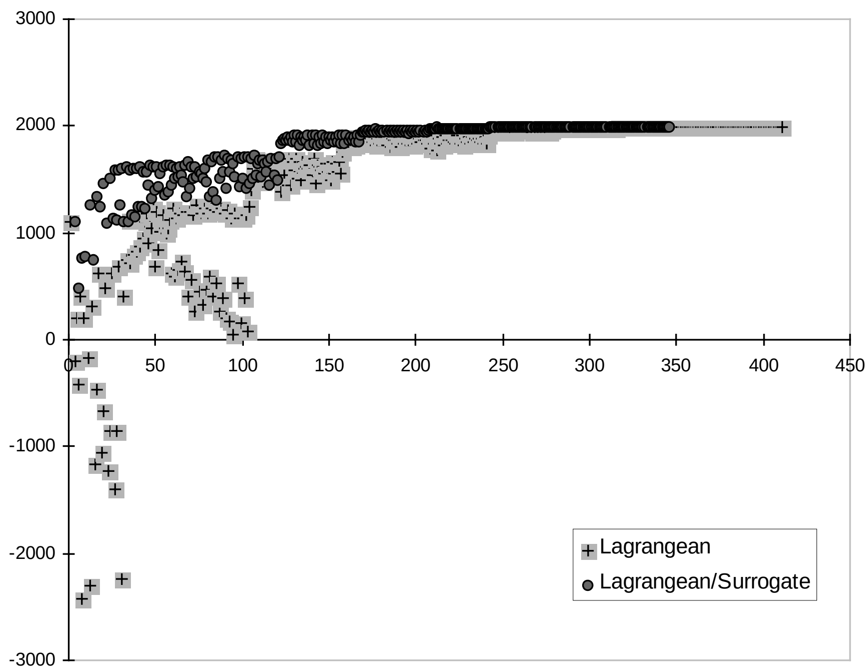

Figure 2. Computational behavior

5. Conclusion

This work considers a Lagrangean/surrogate approach to relaxation heuristics for p-median problems. The Lagrangean/surrogate heuristic was able to generate approximate solutions in a computational time that is about 38% of computational time needed for Lagrangean (alone) heuristic, without loss of quality for duality gaps. We hope that this feature can be explored for even large scale problems to produce high quality approximate solutions at reasonable computational cost. The Lagrangean/surrogate heuristic seems to be better than the ordinary Lagrangean relaxation for all the time consuming instances tested.

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