NONLINEAR DISTRIBUTION OF THE SUNSPOT MAGNETIC FIELD DURING THE SOLAR MAXIMUM

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ABSTRACT

Using the standard solar $\omega$-dynamo equations we formulate appropriate initial and boundary conditions at the moment when the toroidal component is maximum, so that we obtain the toroidal magnetic field component as a function of the solar latitude. We found a new non-homogeneous distribution for the toroidal component along the latitude ($\Delta B_t / \Delta \theta$ given by 7G/degree, 24G/degree and 34G/degree for the respective latitude ranges of 10°-15°, 16°-18° and 19°-25°) and this nonhomogenous decay rate is statistically confirmed through a chi-square analysis of the Mt. Wilson Sunspot data for two solar cycles. Based on this result one can expect the most energetic magnetic eruptive phenomena to be, in average, nonlinearly concentrated at latitudes ranging from 10 to 25 degrees in both hemispheres. © 2003 COSPAR. Published by Elsevier Ltd. All rights reserved.

INTRODUCTION

The magnetic solar cycle lies in the domain of solar dynamo theory which describes a magnetohydrodynamical flow, below the solar surface, capable of sustaining a magnetic field indefinitely against Ohmic decay. In this context, the fields of the active regions on the surface are interpreted as sections of the toroidal dynamo component that float up to form the observed bipolar sunspots regions.

The three mean characteristics of the solar dynamo motions, (i) the differential rotation, (ii) the mean helicity, and (iii) the turbulent diffusivity, mostly determine the mean-field transport, which was demonstrated already in the pioneering dynamo models of the solar magnetoconvection (Parker, 1955, Babcock, 1961, Leighton, 1969, Steenbeck and Krause, 1969). Later on, attention was given also to the turbulent permeability, the meridional circulation, the topological pumping and other features influencing the mean transport, such as absolute instability (Tobias et al., 1997) and new developments based on magnetic helicity conservation (e.g., Brandenburg and Dobler, 2002). Nowadays, the criticisms on solar dynamo model are due theoretical aspects, as the regeneration of the poloidal field from the toroidal one, and observational aspects, as the observation of new magnetic structures (e.g. ephemeral regions, coronal emissions and current lines) (Wilson, 1987). These criticisms have generated improvements of the model (e.g. Schmitt, 1984, Ruzmaikin, 1985, 1991, Choudhuri et al., 1995, Ossendrijver et al., 2002) and the development of new scenarios (e.g. Wilson, 1987, 1991, Covas et al. 1997, Durney, 1997, Subramanian, 2002). Most recent research has concentrated on the regeneration of the poloidal field regime (the so-called $\alpha$-effect). The generation of the toroidal regime from the poloidal one (the so-called $\omega$-effect) is, in fact, studied in less detail because it is not as complex as the $\alpha$-effect. However, there are still some aspects in the transition from the poloidal to the toroidal regime that should be explained by the canonical $\omega$-solar dynamo mechanism. The main question that we address in this paper is, how is the fine distribution of the toroidal magnetic field along solar
latitude at the moment of maximum activity? The answers to this question are important to understand the meridional flow influence on the toroidal field component and consequently its influence on the butterfly diagram fine structure.

In the following section, from an analytical manipulation of ω-dynamo equations, we derive an explicit expression for the toroidal field as a function of the solar latitude when the ω-effect is maximum, and compare the results obtained from this analytical approach to observational data applying a chi-square statistical analysis on the sunspot group data (for two maximum activity periods: Cycles 20 and 21). The final section contains the main implications of our results and our concluding remarks.

MODEL ANALYSIS

A reduced set (the further analysis is independent of angular velocity variation) of the ω-dynamo equations governing the solar cycle [Yoshimura, 1972, 1975, 1976], in spherical coordinates, can be expressed as

\[ \frac{\partial A}{\partial t} = R B \] (1)

\[ \frac{\partial B}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \cos \theta \left( r A + \frac{\partial}{\partial r} A \cos \theta \right) \] (2)

\[ H_\theta = \frac{1}{r} \frac{\partial}{\partial r} \frac{A}{\cos \theta} \] (3)

\[ H_r = \frac{1}{r \cos \theta} \frac{\partial}{\partial \theta} (A \cos \theta) \] (4)

\[ H_\phi = B \] (5)

where \( R \), the regeneration factor, is the rate of the regeneration action of the fluid motion, \( \omega \) is the rotational angular velocity (\( \omega_0 \) is the value of \( \omega \) at the equator, where \( r=R \) and \( \theta = 0 \)), \( A \) is the longitudinal component of the vector potential for the poloidal field, and \( B \) is the intensity of the toroidal magnetic field. The effects of the diffusivity and the meridional circulation are neglected to simplify the problem (Yoshimura, 1975b). Equations (3), (4) and (5) furnish the longitudinal, latitudinal, and radial components of the general axisymmetric field, where \( H_\theta \) and \( H_r \) are the components of the poloidal field.

For the sake of compatibility between an analytical approach, using the ω-dynamo equations, and a statistical analysis of the distribution of the sunspot group magnetic field (next section), we consider the following conditions, (a) at a given instant \( t_0 \) the toroidal component is almost maximum and we have \( H_\phi \neq 0, R \neq 0, A \neq 0, H_r \neq 0 \) and \( H_\theta \to 0 \), (b) at a given instant \( t_1 \) the toroidal component is maximum and we have \( H_\phi \neq 0, R \neq 0, A \neq 0, H_r \neq 0 \) and \( H_\theta = 0 \).

Taking into account \( H_\theta = 0 \), from Eq.(3), we have

\[ A = k \frac{f(\theta)}{r} \] (6)

where \( k \) is a constant of proportionality. From Yoshimura (1975a) we obtain \( f(\theta) = \tan \theta \), so that Eq.(6) becomes \( A = k \frac{\tan \theta}{r} \).

Replacing Eq.(6) into Eq.(2), we find

\[ \frac{\partial B}{\partial t} = -\frac{\partial}{\partial r} \frac{k}{r} \cos \theta \] (7)

where, according to Parker (1987), we have for this model \( \frac{\partial}{\partial r} \frac{k}{r} < 0 \). Integrating Eq.(7), from \( t_0 \) to \( t_1 \), we get

\[ B_{t_1}(\theta, r) - B_{t_0} = -\frac{\partial}{\partial r} \frac{k t_1}{r} \cos \theta \] (8)
From Yoshimura (1975a) we have

\[
\omega(\theta, r) = \omega_0 [1 - 0.2 \sin^2 \theta + \frac{(r - r_s)}{r_s} - 0.2 \frac{(r - r_s)}{r_s} \sin^2 \theta],
\]

(9)

where \(r_s\) is the solar radius and \(\omega_0\) is the angular velocity in the equator (\(\theta = 0, r = r_s\)). Derivating Eq.(9) with respect to \(r\), replacing the result into Eq.(8), and regarding \(r \sim r_s\), we find

\[
B_{t_1, r_s}(\theta) = B_{t_0} + C \cos \theta - 0.2C \cos \theta (1 - \cos^2 \theta),
\]

(10)

where \(C = (k \omega_0 t_1)/r_s^2\) (with \(k = (I \pi r_s^2)/c\)). Addressing the typical values for \(\omega_0 = 3 \times 10^{-6} \text{s}^{-1}\) (Zirin, 1988), \(t_1 \sim 11\text{years} = 3.5 \times 10^8 \text{s}\), \(I = 3 \times 10^8 \text{abA}\) (Alfvén, 1981) and \(c = 3 \times 10^{10} \text{cm.s}^{-1}\), we find \(C = 33\) [c.g.s]. Thus, Eq.(10) now becomes

\[
B_{t_1, r_s}(\theta) = B_{t_0}(\theta) + 33[\cos \theta - 0.2 \cos \theta (1 - \cos^2 \theta)],
\]

(11)

where \(B_{t_0}(\theta)\) is a function of the same form of the second term of Eq.(11). Taking into account a typical value for \(B_{t_0} \sim 1500 \pm 150 \text{G}\) (from the statistics of the 2030 sunspots groups from the Cycles 20 and 21 analysed in the next Section) we obtain, from Eq.(11), a one-hemisphere distribution for the toroidal field \((B \times \theta)\), as shown in Figure 1.

As the length of the toroidal field given by Eq.(11) is connected to the length (and latitudes) of sunspots groups during the solar maximum peak, the numerical results shown in Figures 1 can be compared, in the next Section, to sunspot observational data.

![Fig. 1. The magnetic toroidal field vs. solar latitude for the analytical model given by Eq.(11) (solid line). For comparison purposes, the observational data is also displayed (dashed line). The error bars correspond to the standard deviations given by the Solar Geophysical Data classification for the Mt. Wilson observations.](image)

**SUNSPOTS STATISTICAL ANALYSIS**

A contingency table is usually constructed for the purpose of studying the relationship between two variables. By means of the \(\chi^2\) test it is possible to test the hypothesis that the two variables are independent
(the so-called $H_0$ hypothesis). Thus, in connection with both Table 1 and Table 2, the $\chi^2$ test can be used to check the hypothesis that there is no relationship between an individual sunspot group latitude ($\theta_{SG}$) and its magnetic field ($B_{SG}$).

For a given contingency table, the hypothesis $H_0$ can be tested by means of

$$\chi^2 = \sum_{i=1}^{r} \sum_{j=1}^{c} \frac{(o_{ij} - e_{ij})^2}{e_{ij}},$$  

(12)

where $o_{ij}$ is the observed frequency and $e_{ij}$ is the expected frequency, for a given cell in the $i$th row and $j$th column of a contingency table. Hence, the proper number of degrees of freedom for testing independence in a contingency table of $r$ rows and $c$ columns is given by $\nu = (r - 1)(c - 1)$. The value of expected frequency ($e_{ij}$) in each cell is obtained by multiplying the correspondent total marginals and dividing this product by the total of the sample, $N$. These values, corrected to the nearest integer, are inserted in parentheses in Tables 1 and 2. According to the principles for testing hypotheses the quantity given by Eq.(12) may be treated as possessing a $\chi^2$ distribution with $(r - 1)(c - 1)$ degrees of freedom, provided that $N$ is sufficiently large and $H_0$ is true (Norusis, 1985).

We collected data compatible to the analytical result derived in the second section. The toroidal field in the instant $t_1$ is equivalent to the period of maximum solar activity, when usually the maximum number of sunspots groups appear as a signature of the toroidal field on the photosphere. From the Mt. Wilson Sunspot catalogue (Solar Geophysical Data) we got two samples,

(A) for the period 1975-1986 (Solar Cycle 20), 1614 sunspot groups observed around and at the maximum from 1978 to 1982 (Table 1), and

(B) for the period 1986-1997 (Solar Cycle 21), 416 sunspot groups observed around and at the maximum from 1988 to 1991 (Table 2).

The sunspot group latitude (in degrees), $\theta_{SG}$, is coded as follows, $(0 - 5) = 1$, $(6 - 10) = 2$, $(11 - 15) = 3$, $(16 - 20) = 4$, $(21 - 25) = 5$, $(26 - 30) = 6$, and $(> 30) = 7$. The sunspot group magnetic field, $B_{SG}$ (in Gauss), is coded as follows, $(0 - 500) = 1$, $(600 - 1000) = 2$, $(1100 - 1500) = 3$, and $(> 1500) = 4$. The parameter $B_{SG}$ is the largest magnetic field strength measured in each sunspot group. These measurements are made with the line 5250.216Å(Fe I) and are corrected for projection effects. All values of $B$ were collected just for groups with CMD (Central Meridian Distance) less than 56°. This was a guarantee that the group appeared in the visible disc.

Table 1. Table of Contingency, $B_{SG}$ (first column) and $\theta$ (first row), for the Period Nov 1978-Feb 1982.

<table>
<thead>
<tr>
<th>$B$</th>
<th>$\theta$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9  (9)</td>
<td>17 (24)</td>
<td>20 (32)</td>
<td>26 (30)</td>
<td>34 (19)</td>
<td>12 (8)</td>
<td>8 (5)</td>
<td></td>
<td>126</td>
</tr>
<tr>
<td>2</td>
<td>40 (38)</td>
<td>106 (105)</td>
<td>127 (141)</td>
<td>128 (131)</td>
<td>92 (83)</td>
<td>33 (35)</td>
<td>31 (22)</td>
<td></td>
<td>557</td>
</tr>
<tr>
<td>3</td>
<td>34 (44)</td>
<td>119 (121)</td>
<td>185 (163)</td>
<td>160 (151)</td>
<td>81 (95)</td>
<td>45 (41)</td>
<td>16 (25)</td>
<td></td>
<td>640</td>
</tr>
<tr>
<td>4</td>
<td>28 (20)</td>
<td>63 (55)</td>
<td>78 (74)</td>
<td>67 (69)</td>
<td>33 (43)</td>
<td>13 (19)</td>
<td>9 (11)</td>
<td></td>
<td>291</td>
</tr>
<tr>
<td>Total</td>
<td>111</td>
<td>305</td>
<td>410</td>
<td>381</td>
<td>240</td>
<td>103</td>
<td>64</td>
<td></td>
<td>$N = 1614$</td>
</tr>
</tbody>
</table>

Table 2. Table of Contingency, for the Period Nov 1988-Jul 1991.

<table>
<thead>
<tr>
<th>$B$</th>
<th>$\theta$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>42 (42.8)</td>
<td>61 (61.6)</td>
<td>253 (251.6)</td>
<td></td>
<td>356</td>
</tr>
<tr>
<td>4</td>
<td>8 (7.2)</td>
<td>11 (10.4)</td>
<td>41 (42.4)</td>
<td></td>
<td>60</td>
</tr>
<tr>
<td>Total</td>
<td>50</td>
<td>72</td>
<td>294</td>
<td></td>
<td>$N = 416$</td>
</tr>
</tbody>
</table>

For the data at Table 1 we found that, by using Eq.(12), $\chi^2 = 49.8$. Since $\chi^2_T = 34.8$ for $\nu = 18$ degrees of freedom (with significance level equals 0.01), this result is significant and the hypothesis $H_0$ of independence is therefore rejected.
For the data at Table 2 we found that, by using Eq. (12), $\chi^2 = 0.2$. Since $\chi^2 = 9.2$ for $\nu = 2$ degrees of freedom (with significance level equals 0.01), this result is significant and the hypothesis $H_0$ of independence is therefore rejected.

Assuming a toroidal field configuration symmetrically distributed at both solar hemispheres, we compare the numerical results obtained from Eq. (11) against observational data introduced in this section. Considering the statistical deviation of the observational data, Figure 1 shows a reasonable agreement between them, mainly in terms of the qualitative nonlinear decay of $B(\theta)$ against $\theta$.

CONCLUDING REMARKS

From the statistics of the 2030 sunspots groups analysed from the Cycles 20 and 21 we found that $\sim 80\%$ of the sunspots groups with $B_{SG} > 10^3 G$ are inside the latitude range of $10^\circ - 25^\circ$. Moreover, the statistical data analysis in tables of contingency 1 and 2 shows that there is a statistical dependence between the parameters $B_{SG}$ and $\theta_{SG}$, in a way that sunspots groups, in the toroidal region, emerge with more intense magnetic field at low latitudes than those in high latitudes. Note that this is the same tendency showed by the global butterfly diagram, here partially characterized inside the toroidal region at the peak of the solar maximum. If the latter works in the entire convection zone the $\omega$-dynamo alone brings information on the meridional flow and implies toroidal field belts migrating equatorwards. Therefore, the Butterfly diagram can be due to meridional motions as given by the complete $\alpha - \omega$-dynamo solutions considering the meridional circulation (e.g. Dikpati and Gilman, 2001, Bonanno et al, 2002, and Rudiger and Elstner, 2002).

The theoretical analysis at Section 1 shows that (if the classical $\omega - dynamo$ works, at least, at the poloidal-toroidal stage) this deterministic spatial distribution, of magnetic field, is also spatially nonhomogeneous with the rates of $-\frac{\partial H}{\partial \theta}$ given by $7G/degree$, $24G/degree$ and $34G/degree$ for the respective latitudes ranges of $10^\circ-15^\circ$, $16^\circ-18^\circ$ and $19^\circ-25^\circ$. Characterization of this fine nonlinear distribution is important to understand the geometric complexity of the spatial distribution of magnetic fields on the solar surface. The connection with self-similarity as reported by Tao et al. (1995) should be dynamically investigated as a source of possible long-range spatio-temporal correlations driving the sunspots emergence mechanism.

An immediate implication of this result is about observational logistic concerning solar eruptive phenomena prediction. Once most of solar flare energy comes from the magnetic structures as coronal loops, one can expect the most energetic eruptive phenomena to be, in average, concentrated at latitudes $10^\circ \leq \theta \leq 25^\circ$. The energetic counterpart signature of this nonlinear magnetic field distribution must be checked also at others frequencies in the solar atmosphere, preferentially for spatial high resolution data ($\leq 2^\circ$) from X-rays, UV and radio observations.

Finally, we expect this result be approximately robust for higher intensity magnetic field as the values reported by Mac Gregor and Charbonneau (1997) and also compatible with the recent unstable differential rotation model suggested by Dikpati and Gilman (1999). Using total magnetic flux measurements carried out on Kitt Peak and MDI magnetograms, we may obtain a better fitting between the theoretical curve given by Eq. (11) and the observational data. Work along these subjects is in progress and will be communicated later.

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